

Ordered Pairs, Relations

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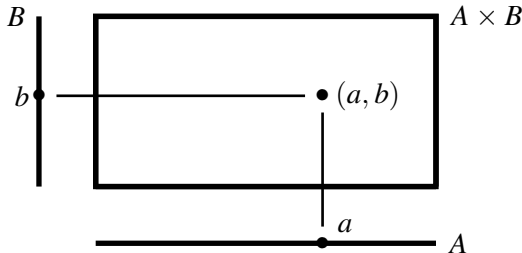
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John			*	
Paul	*	*	*	*
George	*	*	*	
Ringo			*	*

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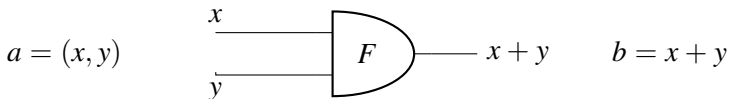
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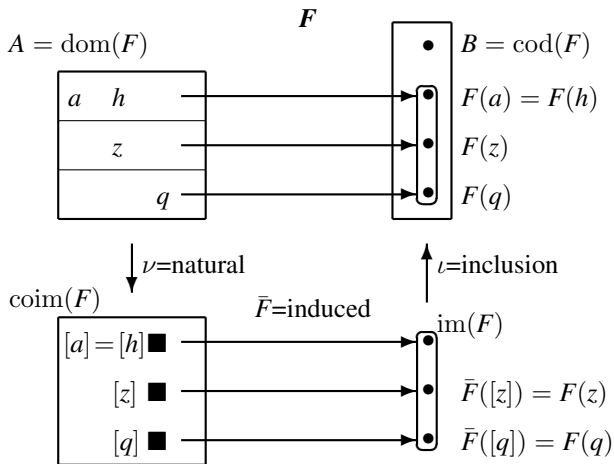
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- The **canonical factorization of F** is $F = \iota \circ \bar{F} \circ \nu$.