

The Natural Numbers

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This axiom guarantees that $\mathbb{N}$ exists.

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- ① $\mathbb{N} = \{0, 1, 2, \dots\}$. ✗
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Why did we define $\mathbb{N}$ this way?

There are different inductive sets, like

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and the hope is that if we intersect all of them we will be left with only

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