Naive Set Theory versus Axiomatic Set Theory

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More precisely, we will show that if we allow the construction principle of unrestricted comprehension, then some statements of the form "$x \in y$" are neither true nor false.

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But formally, we just consider x to be "a thing", like a dot on the page, and $x \in y$ asserts that x and y are related in some way. We may write this relation as $x \to y$ or $\bullet \to \bullet$ or $\overset{x}{\bullet}\!\to\!\overset{y}{\bullet}$.

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Formal set theory is founded on axioms, which restrict the possible meanings of "set" ($x, y, z, \ldots$) and "membership" ($\in$).
The axioms were chosen to reflect our intuition about "unordered collections of distinct objects".

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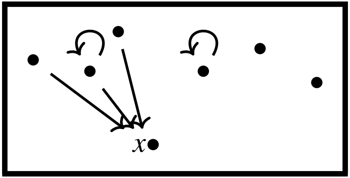

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This is just a collection of some of the vertices in D. Is C a set? For this, we would need some vertex v in D whose collection of in-neighbors is exactly C. Then v "names" the collection, and this process of "naming" certifies that C is a set.

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Question: Is R a set? That is, is there a vertex v whose set of in-neighbors is exactly R? (Key issue: If such a v exists, does it have a loop?)

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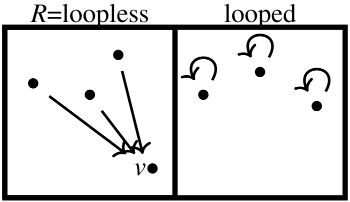

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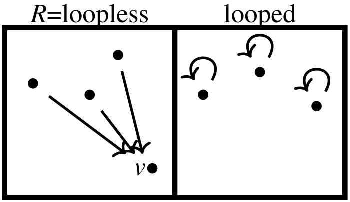

Case 2:

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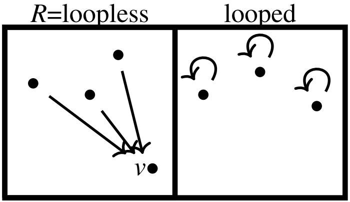

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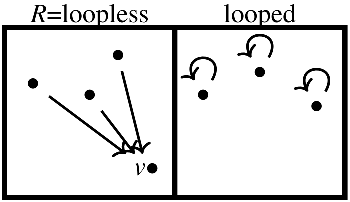

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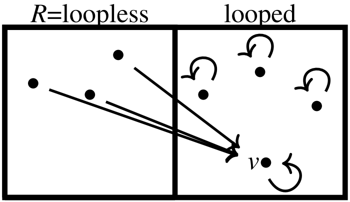

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Corollary.

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Case 2. $R \in R$.
If $R \in R$, then R fails the defining formula "$x \notin x$". Since R fails the definition for membership in R, we derive that $R \notin R$, a contradiction. $\square$

Corollary. Naive set theory is inconsistent. $\square$