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For an alternative proof, use the formula at the top of the page.

Pascal's Triangle as an $n \times k$ table

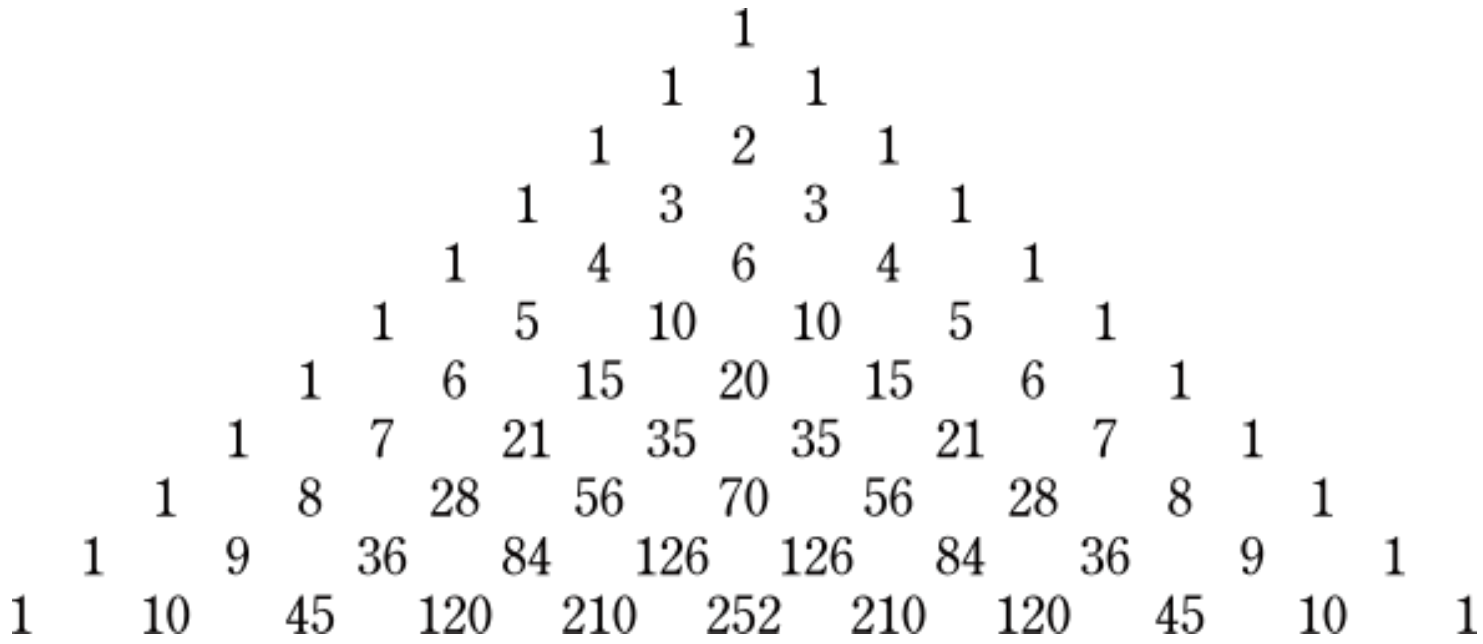
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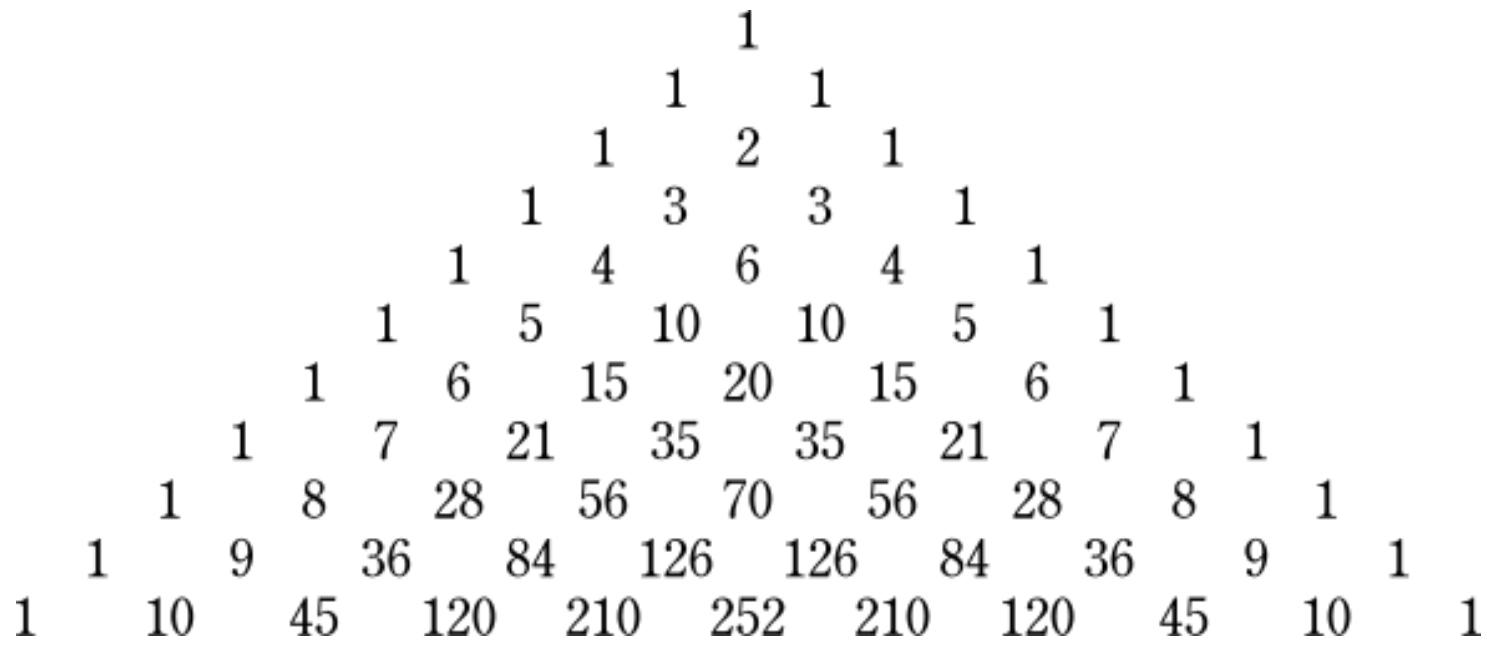
n \ $\binom{n}{k}$	0	1	2	3	4	5	6	$\cdots$
0	1	0	0	0	0	0	0	
1	1	1	0	0	0	0	0	
2	1	2	1	0	0	0	0	
3	1	3	3	1	0	0	0	
4	1	4	6	4	1	0	0	
5	1	5	10	10	5	1	0	
6	1	6	15	20	15	6	1	
$\vdots$								$\ddots$

Pascal's Triangle as a triangle

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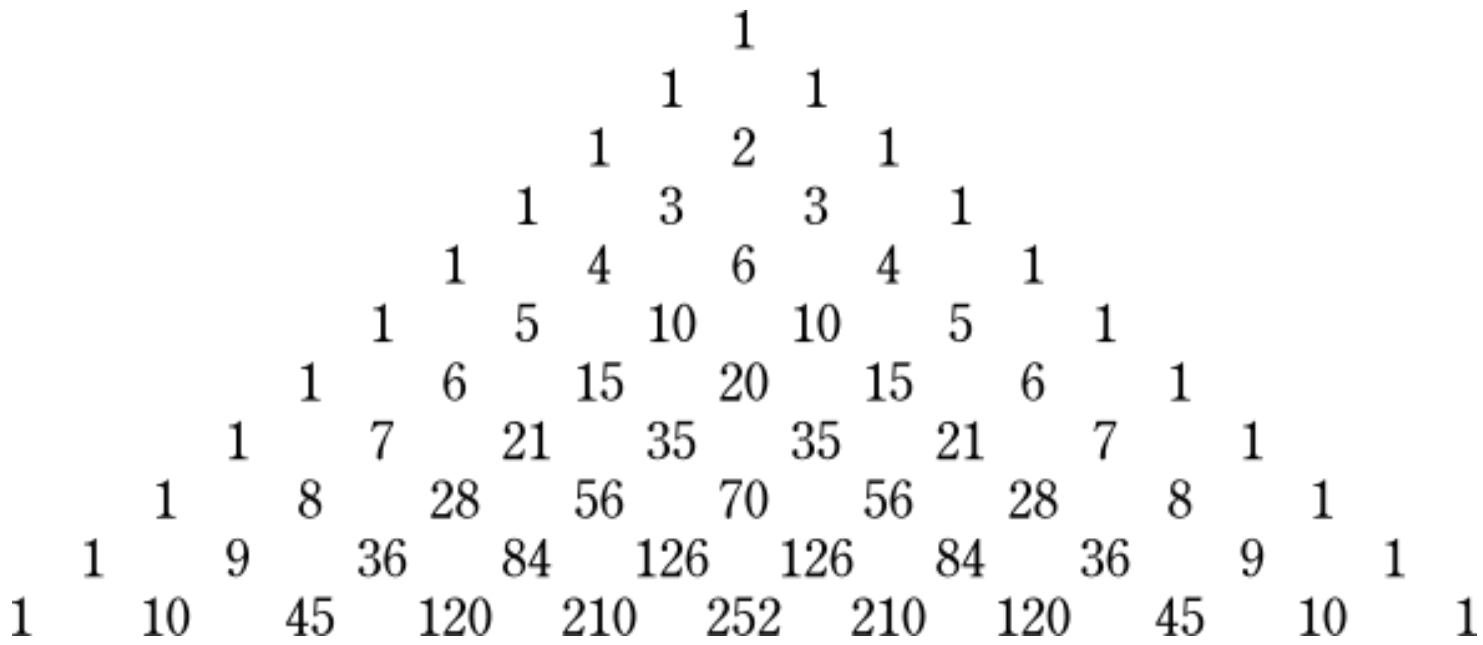


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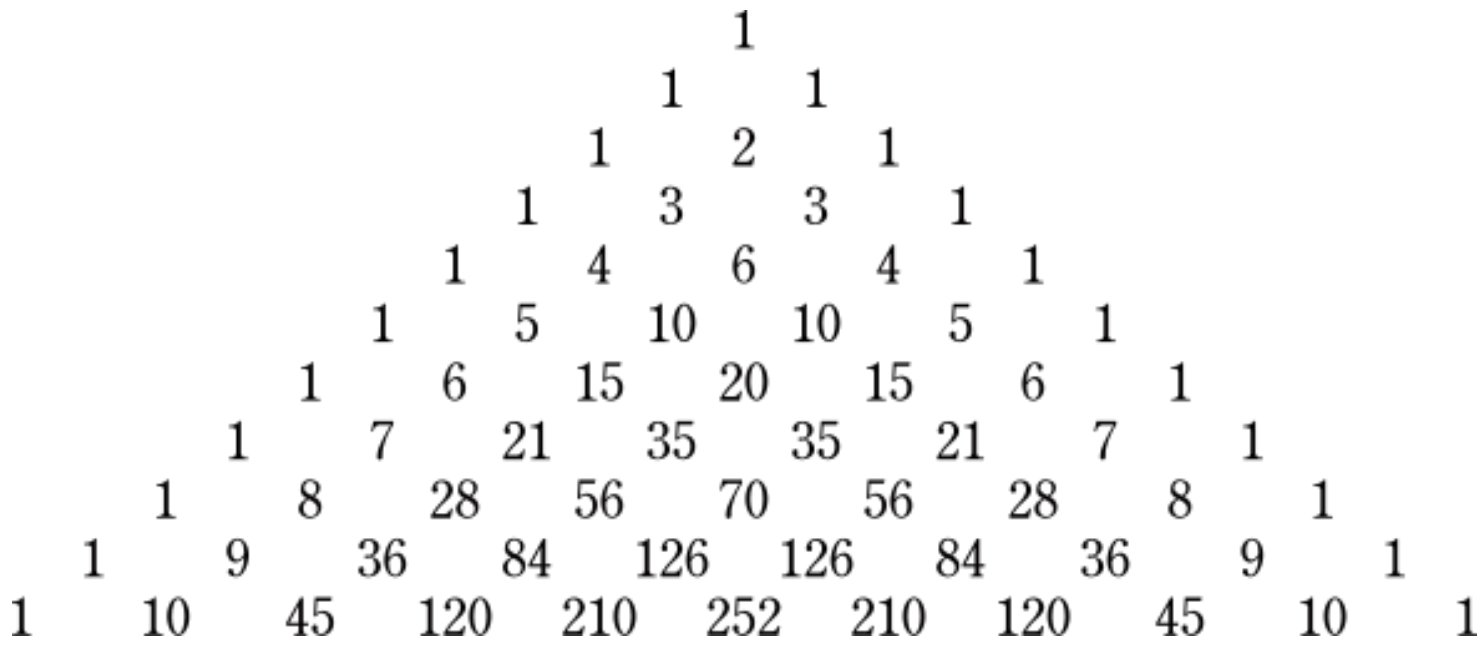
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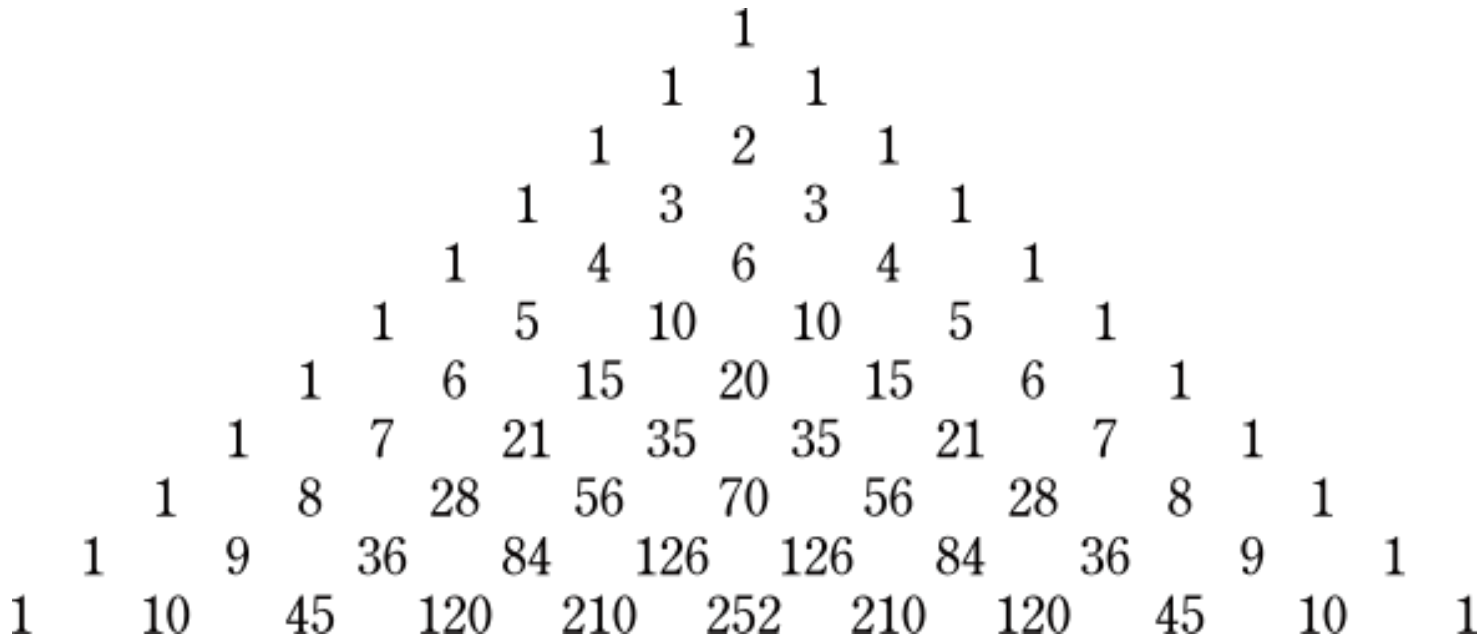
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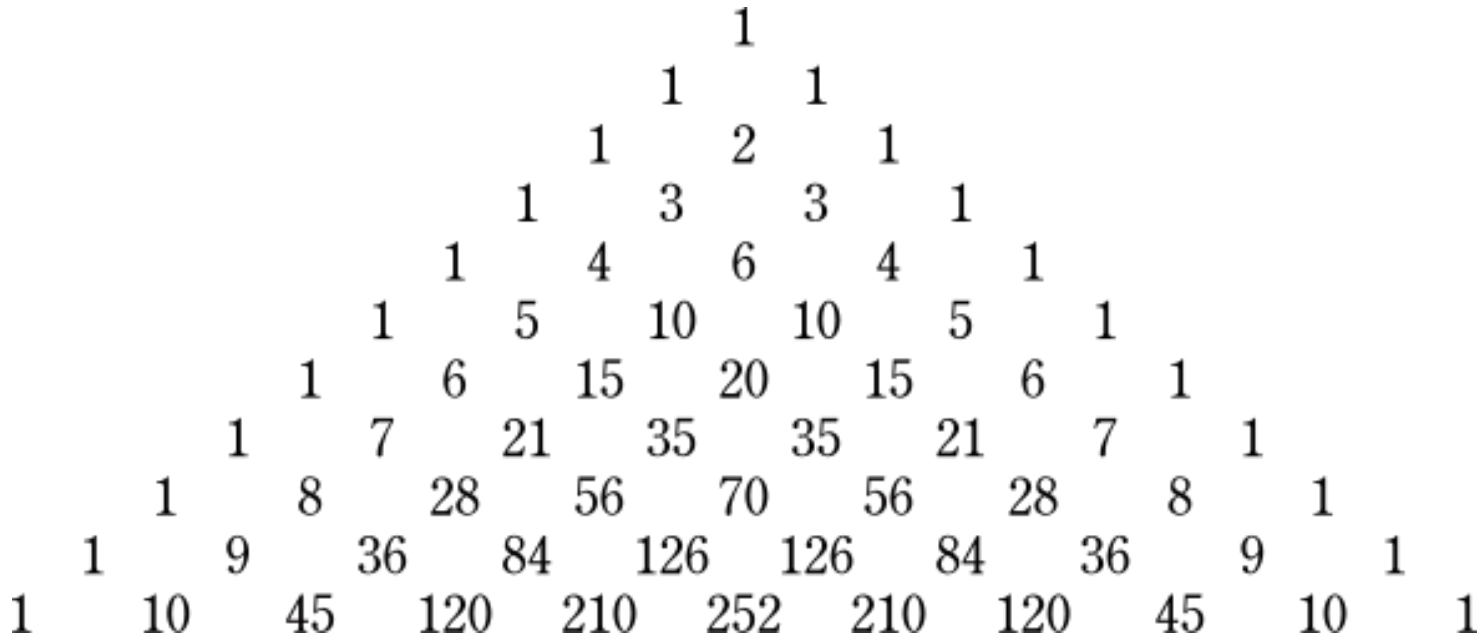
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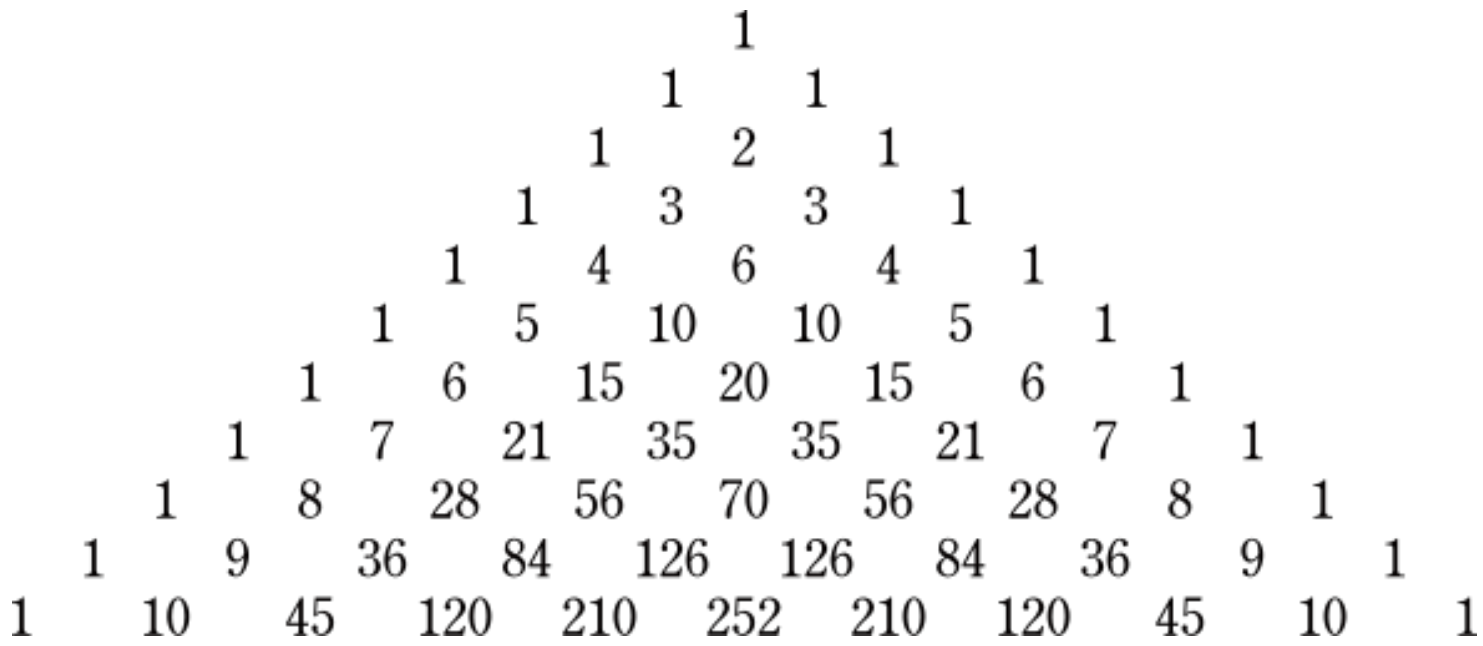
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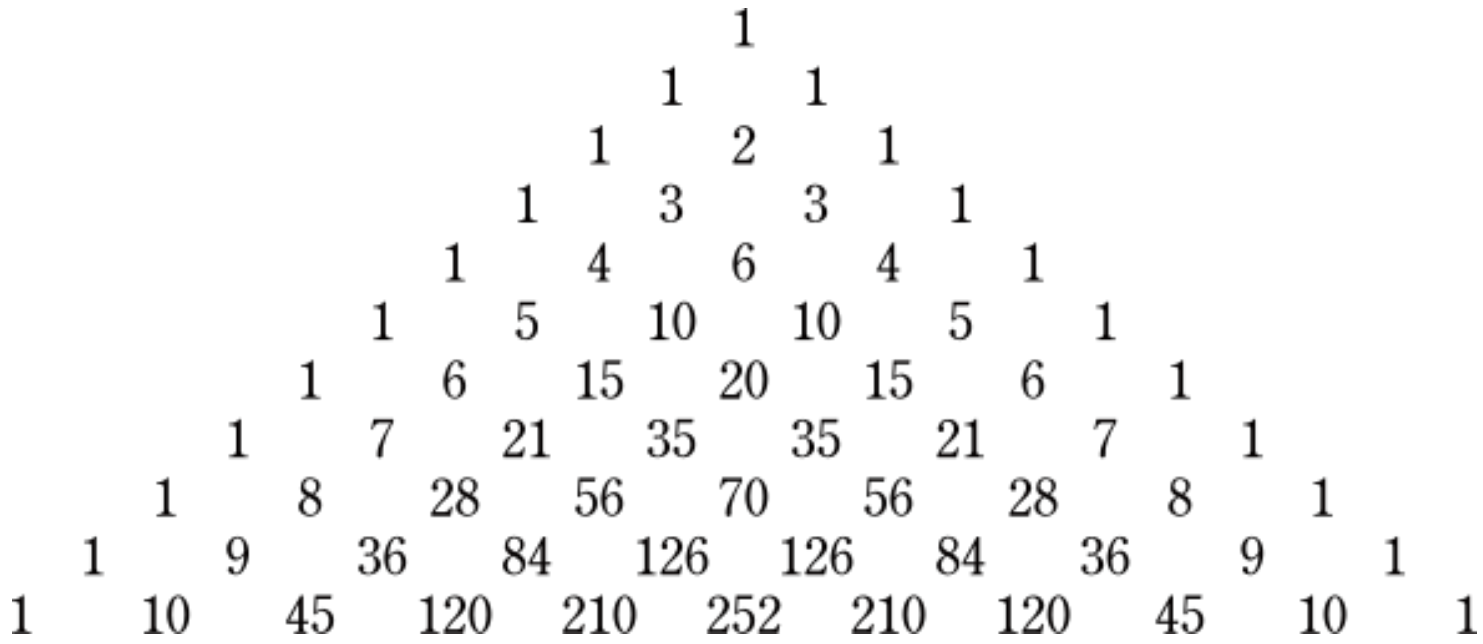
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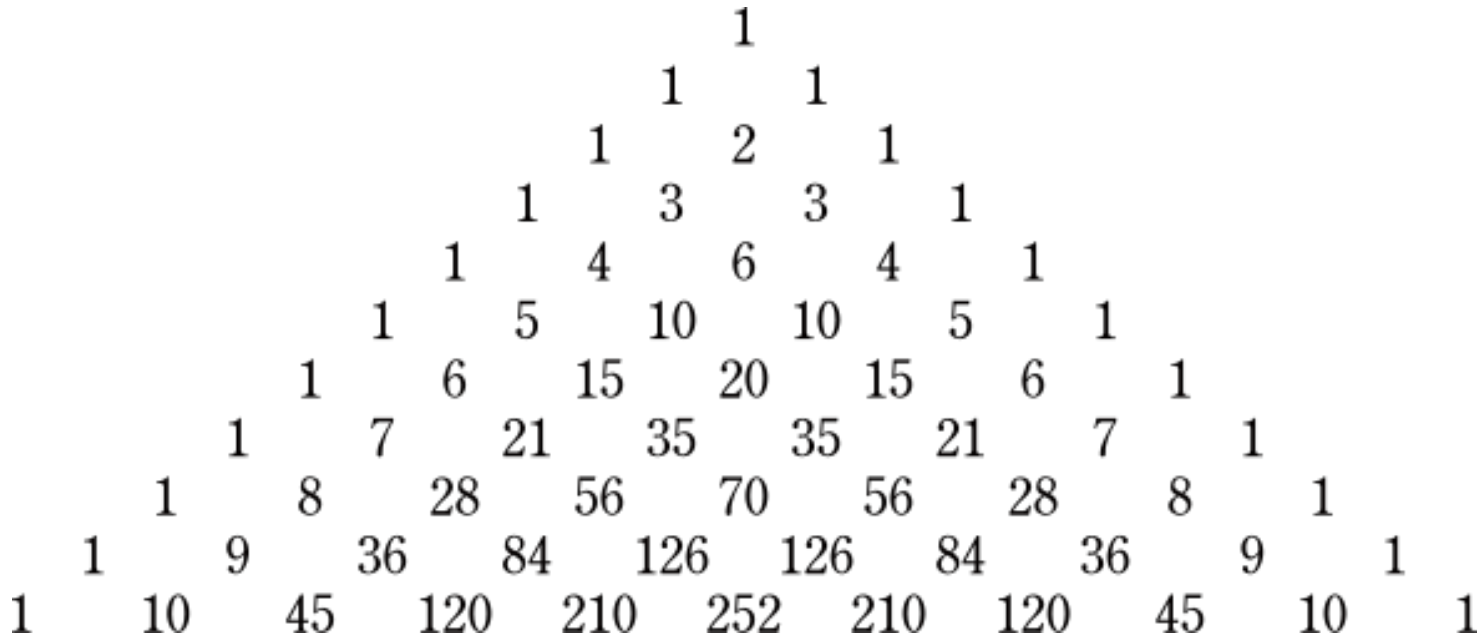
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The Binomial Theorem

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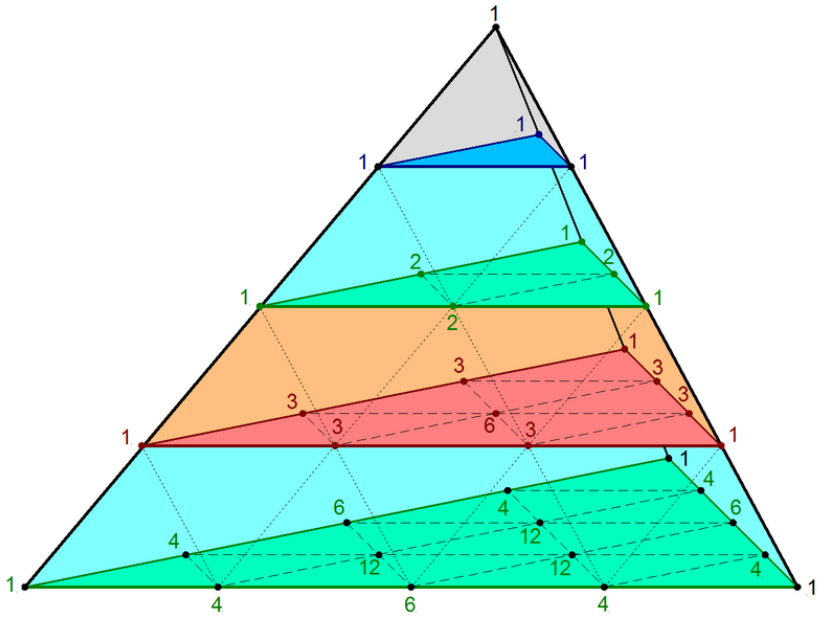
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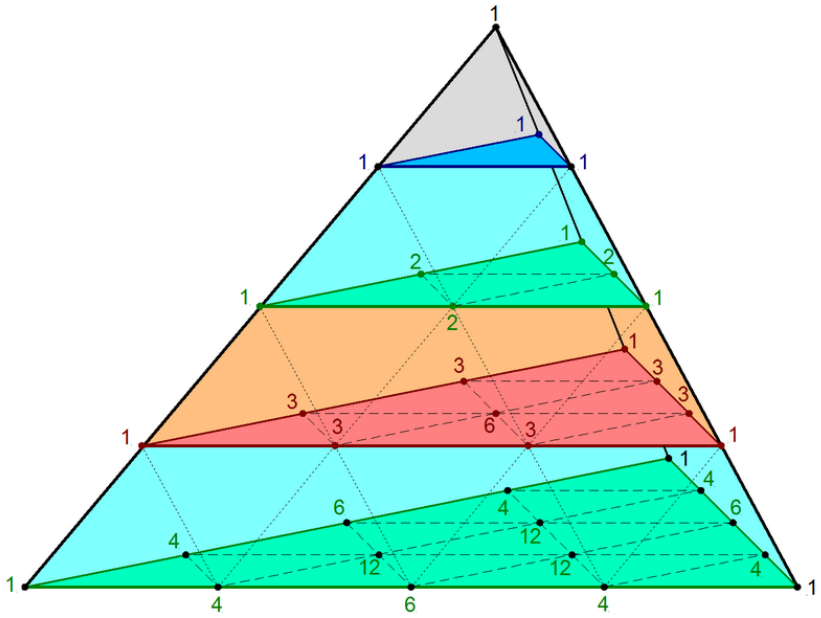
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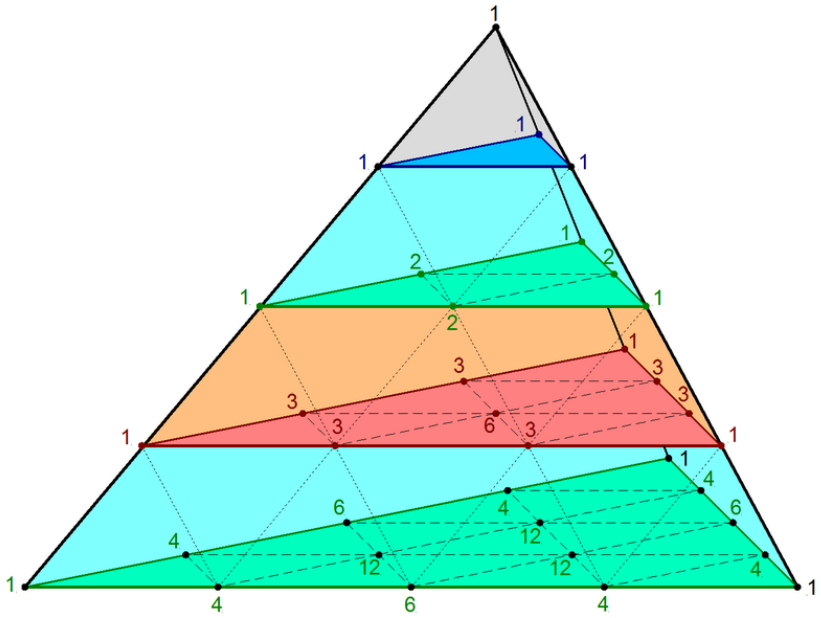


Pascal's Pyramid



Multinomial Theorem.

Pascal's Pyramid



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$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+x_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

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