The Completeness Theorem, Part 1
Metatheorems

Metatheorems for preparing Γ

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Our argument assumed the truth of the theorem called "Generalization on Constants", (Theorem 24F), and its Corollary, (Corollary 24G). Let's justify those assumptions now.

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Corollary 24G. Assume that $\Gamma \vdash \varphi^x_c$ where c is a constant that does not occur in $\Gamma \cup \{\varphi\}$.

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Corollary 24G. Assume that $\Gamma \vdash \varphi^x_c$ where c is a constant that does not occur in $\Gamma \cup \{\varphi\}$. There is a Γ-proof of $(\forall x)\varphi$ in which c does not occur.

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Proof. To see that $(\ddagger)$ is a Γ-proof, we first show that each $(\alpha_i)^c_y$ in the sequence is an element of Γ, an axiom, or follows from earlier terms using MP.

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1. If $\alpha_i \in \Gamma$, then since c does not appear in Γ we have $(\alpha_i)^c_y = \alpha_i \in \Gamma$.
2. If α_k follows from α_i and $\alpha_j = (\alpha_i \to \alpha_k)$, $i, j < k$, then in $(\ddagger)$ we will have $(\alpha_j)^c_y = ((\alpha_i)^c_y \to (\alpha_k)^c_y)$, so $(\alpha_k)^c_y$ follows from earlier terms in $(\ddagger)$ by MP.

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3. We now examine that case where α_i is an axiom from one of the six axiom familes.

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1. (α_i is a tautology) ✓
2. (α_i is $(\forall x)\varphi \to \varphi^x_t$, t substitutable)

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4. (α_i is $\varphi \to (\forall x)\varphi$), x not free in φ)

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Remaining Goal: show that if α_i is an axiom, and y does not appear in α_i, then $(\alpha_i)^c_y$ is also an axiom.

1. (α_i is a tautology) ✓
2. (α_i is $(\forall x)\varphi \to \varphi^x_t$, t substitutable) ✓

$$(\alpha_i)^c_y \quad \text{is} \quad ((\forall x)\varphi \to \varphi^x_t)^c_y \quad \text{is} \quad (\forall x)\varphi^c_y \to (\varphi^c_y)^x_{t^c_y}$$

3. (α_i is $(\forall x)(\psi \to \varphi) \to ((\forall x)\psi \to (\forall x)\varphi)$) ✓

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First Metatheorem, Part 3

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is a Γ-proof of φ^c_y in which c does not occur. Let $\Phi \subseteq \Gamma$ be the finite set of elements of Γ used in the deduction $(\ddagger^-)$. Since y does not appear (free) in Φ, we may use Generalization to continue this proof to deduce $(\forall y)\varphi^c_y$.

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Second Metatheorem

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Corollary 24G.

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Proof:

Second Metatheorem

Corollary 24G. Assume that $\Gamma \vdash \varphi_c^x$ where c is a constant that does not occur in $\Gamma \cup \{\varphi\}$. There is a Γ-proof of $(\forall x)\varphi$ in which c does not occur.

Proof: By Generalization on Constants,

Second Metatheorem

Corollary 24G. Assume that $\Gamma \vdash \varphi_c^x$ where c is a constant that does not occur in $\Gamma \cup \{\varphi\}$. There is a Γ-proof of $(\forall x)\varphi$ in which c does not occur.

Proof: By Generalization on Constants, the fact that $\Gamma \vdash \varphi_c^x$ where c does not appear in Γ or in φ implies that there is a Γ-proof of $(\forall y)((\varphi_c^x)_y^c)$ in which c does not occur

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Exercise 9, Section 2.4(a)

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Exercise 9, Section 2.4(a)

(a) Show by example that $(\varphi_y^x)_x^y$ is not in general equal to φ.

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Corollary 24G. Assume that $\Gamma \vdash \varphi_c^x$ where c is a constant that does not occur in $\Gamma \cup \{\varphi\}$. There is a Γ-proof of $(\forall x)\varphi$ in which c does not occur.

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Exercise 9, Section 2.4(a)

(a) Show by example that $(\varphi_y^x)_x^y$ is not in general equal to φ.

(b) Show that if y does not occur at all in φ, then $(\varphi_y^x)_x^y = \varphi$.