

FOUNDATIONS OF MATH MIDTERM

You have 50 minutes for this exam. You may not use any unauthorized sources, and you may not communicate with others about the exam. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

There are four problems on this test, each with two parts. Problems 1 and 2 concern sentential/propositional logic, while Problems 3 and 4 concern first-order logic. Select one problem from propositional logic (either Problem 1 or Problem 2) and complete both parts of that problem. Then select one problem from first-order logic (either Problem 3 or Problem 4) and complete both parts of that problem. Please circle the problem numbers of the problems that you want to have graded.

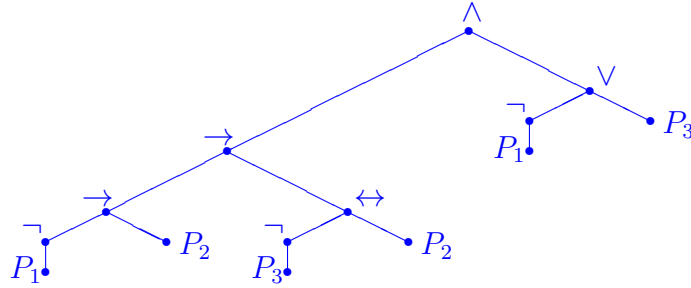
I have neither given nor received aid on this exam.

Name: _____

1. Let α be the propositional formula

$$(((\neg P_1) \rightarrow P_2) \rightarrow ((\neg P_3) \leftrightarrow P_2)) \wedge ((\neg P_1) \vee P_3).$$

(a) Draw a formula tree for α .



(b) Write down a truth table for α .

P_1	P_2	P_3	$\beta = ((\neg P_1) \rightarrow P_2)$	$\gamma = ((\neg P_3) \rightarrow P_2)$	$(\beta \rightarrow \gamma)$	$((\neg P_1) \vee P_3)$	α
T	T	T	T	F	F	T	F
T	T	F	T	T	T	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	F	F
F	T	T	T	F	F	T	F
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	T	T	T

2. A set Σ of propositional formulas is **independent** if $\Sigma - \{\alpha\} \not\models \alpha$ for all $\alpha \in \Sigma$. (In words, Σ is independent if no formula $\alpha \in \Sigma$ is tautologically implied by the remaining formulas in Σ .)

For parts (a) and (b) of this problem, let $\Sigma = \{P_1, P_2, P_3, \dots\}$ be the set of all propositional variables.

(a) Show that Σ is independent.

Choose any $P_i \in \Sigma$. The valuation defined by $v(P_k) = \begin{cases} T & \text{if } k \neq i, \text{ and} \\ F & k = i \end{cases}$

witnesses that $\Sigma - \{P_i\} \not\models P_i$.

(b) Show that if β is a formula that is not a sentence symbol, then $\Sigma \cup \{\beta\}$ is not independent.

To argue toward a contradiction, assume that $\Sigma \cup \{\beta\}$ is independent. Then $\Sigma \not\models \beta$, so there is a valuation v_β such that $v_\beta(P_i) = T$ for all i and $v_\beta(\beta) = F$. Choose P_k that does not appear in β . There must exist a valuation v_k witnessing that $(\Sigma - \{P_k\}) \cup \{\beta\} \not\models P_k$. Such a valuation must satisfy $v_k(\beta) = T$, $v_k(P_i) = T$ if $i \neq k$, and $v_k(P_k) = F$. There can be no such valuation, since v_β and v_k agree on the sentence symbols in β , but do not agree on β .

3. Let L be a first-order language with one unary relation symbol P .

(a) Show that $(\forall x P(x) \rightarrow \exists x P(x))$ is a valid sentence.

Suppose our variable set is $V = \{x_0, x_1, \dots\}$ and $x = x_0$. It suffices to show that if v is a valuation in some structure $\mathbf{A}$ and $(\forall x_0)P(x_0)[v] = T$, then $(\exists x_0)P(x_0)[v] = T$.

We know that $(\forall x_0)P(x_0)[v] = T$ if and only if $P(x_0)[v'] = T$ for every valuation $\overline{\{0\}}$ -equivalent to v . Any such v' is witness that $(\exists x_0)P(x_0)[v] = T$.

(b) Show that $(\exists x P(x) \rightarrow \forall x P(x))$ is not a valid sentence.

Assume that $\langle \mathbb{R}; P(x) \rangle$ is the structure with universe $\mathbb{R}$ and predicate P defining positivity ($P(r) = T$ if $r \in \mathbb{R}$ is positive and false otherwise). The given sentence does not hold in this structure, since $(\exists x)P(x)$ is true in the structure, $(\forall x)P(x)$ is false in the structure, and $((\text{True}) \rightarrow (\text{False}))$ is False.

4. Let $\mathbf{A} = \langle [0, 1]; \cdot \rangle$ be the unit interval of the real line equipped with the usual multiplication.

(a) Explain why $f(x) = x^r$ is an automorphism of $\mathbf{A}$ for any real $r > 0$.

If $x \in [0, 1]$, then $x^r \in [0, 1]$ for $r > 0$. The function $x \mapsto x^r$ has an inverse of a similar form, namely $x \mapsto x^{\frac{1}{r}}$. This makes $f(x) = x^r$ a permutation of $[0, 1]$ with inverse of a similar form $f^{-1}(x) = x^{\frac{1}{r}}$. Any function of this form preserves multiplication, since $f(x \cdot y) = (x \cdot y)^r = x^r \cdot y^r = f(x) \cdot f(y)$. The inverse also preserves multiplication by the same argument, so f is an automorphism.

(b) Explain why

(i) $\{0\}$ is definable in $\mathbf{A}$. $\{1\}$ is definable in $\mathbf{A}$.

(ii) If U is a definable subset of $[0, 1]$ and $U \cap (0, 1) \neq \emptyset$, then $(0, 1) \subseteq U$. Hence $U = (0, 1), (0, 1], [0, 1)$, or $[0, 1]$.

For (i), define $\{0\}$ by $\varphi_0(x) : (\forall y)(x \cdot y = x)$. Define $\{1\}$ by $\varphi_1(x) : (\forall y)(x \cdot y = y)$.

For (ii), if U is definable and u belongs to $U \cap (0, 1)$, then $u^r \in U$ for all $r > 0$. But if $u \in (0, 1)$, then every $v \in (0, 1)$ is of the form u^r for some $r > 0$, namely for $r = \log_u(v) > 0$. Thus, $U \cap (0, 1) \neq \emptyset$ implies $(0, 1) \subseteq U$. For the last sentence, any $U \subseteq [0, 1]$ that contains $(0, 1)$ must be $U = (0, 1), (0, 1], [0, 1)$, or $[0, 1]$.