Solutions to HW 8.

2. (Exercise 6(a) of Section 2.4.) Show that if $\vdash \alpha \to \beta$, then $\vdash (\forall x)\alpha \to (\forall x)\beta$.

Suppose that we have a proof of $\alpha \to \beta$ from $\Gamma = \emptyset$:

$$\gamma_0, \gamma_1, \ldots, \gamma_n = \alpha \to \beta.$$

Let's continue it with the following formulas to prove $(\forall x)\alpha \to (\forall x)\beta$:

$\alpha \to \beta$,	End of previous proof
$(\forall x)(\alpha \to \beta)$,	Generalization
$(\forall x)(\alpha \to \beta) \to ((\forall x)\alpha \to (\forall x)\beta)$,	$\forall$-Distribution
$(\forall x)\alpha \to (\forall x)\beta$.	MP

3. (Exercise 6(b) of Section 2.4.) Show that it is not in general true that $\alpha \to \beta \vdash (\forall x)\alpha \to (\forall x)\beta$.

Let α be any valid formula, e.g. a tautology. Then $\alpha \to \beta$ is propositionally equivalent to β and $(\forall x)\alpha \to (\forall x)\beta$ is propositionally equivalent to $(\forall x)\beta$. With such a choice for α, we can reduce the problem to the problem of exhibiting β such that $\beta \nvdash (\forall x)\beta$.

Let β be the formula "$P(x)$" for a unary predicate P. Let $\mathbf{A} = \langle \mathbb{R}; P(x) \rangle$ be the structure where $P(x)$ holds for x iff x is positive. Let v be any valuation in $\mathbf{A}$ for which $v(x) = 1$. Then $\beta[v] = T$, since $P(1)$ is true in $\mathbb{R}$ (since 1 is positive). But $(\forall x)\beta[v] = F$, since $(\forall x)P(x)$ is not true in $\mathbb{R}$. (Not all reals are positive.) This shows that $\beta \nvDash (\forall x)\beta$, hence $\beta \nvdash (\forall x)\beta$.