

Solutions to HW 7.

2. (Exercise 27 of Section 2.2.) List all of the non-isomorphic structures of size 2 in the language whose only nonlogical symbol is one 2-place predicate symbol P .

Every 2-element structure in this language is isomorphic to one of the form $\mathbf{A} = \langle A; P(x, y) \rangle$ where A is the specific 2-element set $\{a, b\}$. To determine $\mathbf{A}$ up to isomorphism, we need only specify the table for $P^{\mathbf{A}}$. There are 10 possibilities for this table, which are:

$P^{\mathbf{A}}$	a	b
a	F	F
b	F	F

$P^{\mathbf{A}}$	a	b
a	T	F
b	F	F

$P^{\mathbf{A}}$	a	b
a	T	F
b	F	T

$P^{\mathbf{A}}$	a	b
a	F	T
b	F	F

$P^{\mathbf{A}}$	a	b
a	T	T
b	F	F

$P^{\mathbf{A}}$	a	b
a	F	T
b	F	T

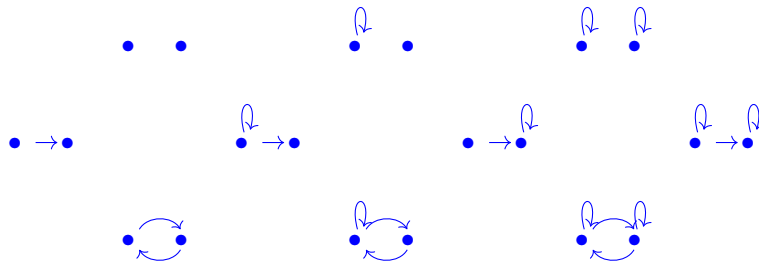
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Such a structure may be viewed as a directed graph with edge set $A = \{a, b\}$ and adjacency relation $P(x, y)$. With this viewpoint, the question asks you to list all non-isomorphic 2-element directed graphs. We can draw pictures of the different isomorphism types, as follows:



Discussion:

If $\mathbf{B} = \langle \{u, v\}; P^{\mathbf{B}} \rangle$ is any 2-element structure in this language and $\beta: \{a, b\} \rightarrow \{u, v\}$ is a bijection, then there is a unique way to define a table for $P^{\mathbf{A}}$ to make β an isomorphism from $\mathbf{A} := \langle \{a, b\}; P^{\mathbf{A}} \rangle$ to $\mathbf{B} = \langle \{u, v\}; P^{\mathbf{B}} \rangle$. Namely, define $P^{\mathbf{A}}(x, y) := P^{\mathbf{B}}(\beta(x), \beta(y))$. This shows that every 2-element structure in this language is isomorphic to one with universe $A = \{a, b\}$. Hence, we only need to count the number of isomorphism types with this universe.

There are $2^4 = 16$ ways to fill out the 2×2 table for P , so there are at most 16 isomorphism types. If $\mathbf{A}_1 = \langle \{a, b\}; P^{\mathbf{A}_1} \rangle$ and $\mathbf{A}_2 = \langle \{a, b\}; P^{\mathbf{A}_2} \rangle$ are two different but isomorphic structures, then it must be that the function $\gamma: \mathbf{A}_1 \rightarrow \mathbf{A}_2: a \mapsto b, b \mapsto a$ that switches a and b is the (unique)

isomorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$. When this exists, say that $\mathbf{A}_1$ and $\mathbf{A}_2$ are “dual” to each other. The relationship between there tables is the following: If the table for $P^{\mathbf{A}_1}$ has the shape on the left, then the dual table for $P^{\mathbf{A}_2}$ has the shape on the right:

$P^{\mathbf{A}_1}$	a	b
a	I	J
b	K	L

$P^{\mathbf{A}_2}$	a	b
a	L	K
b	J	I

,

$I, J, K, L \in \{T, F\}$. $\mathbf{A}$ will be its own dual iff its P -table has the shape:

$P^{\mathbf{A}_1}$	a	b
a	I	J
b	J	I

(That is, $I = L, J = K$ and $I, J \in \{T, F\}$.) There are $2^2 = 4$ tables with this shape.

To count the total number of isomorphism types, restrict the isomorphism relation to the 16 different structures on the set $\{a, b\}$. This relation partitions the 16 structures into some 2-element cells of the form $\{\mathbf{structure}, \mathbf{dual\ structure}\}$ and the rest are 1-element cells of the form $\{\mathbf{self - dual\ structure}\}$. There are $2^2 = 4$ self-dual structures, so there will be 4 cells that have only one member. The remaining $16 - 4 = 12$ structures fall into $\frac{1}{2}(16 - 4) = 6$ cells of size 2, each containing one structure and its isomorphic dual. Altogether, this yields $4 + 6 = 10$ isomorphism types of 2-element structures in this language.

3. Give an example of a universal sentence and an embedding, where the embedding does not preserve satisfaction of the universal sentence. Then, give an example of an existential sentence and an embedding, where the embedding does not reflect satisfaction of the existential sentence.

Our example sentence will be written in the language of one unary predicate, $P(x)$. We express the idea that “every element satisfies P ” with a sentence σ defined by

$$\sigma := “(\forall x)P(x)”.$$

Now let $\mathbf{B} = \langle \mathbb{R}; P(x) \rangle$ where for $r \in \mathbb{R}$ we make the choice that r satisfies P iff r is positive. Let $\mathbf{A} = \langle \mathbb{R}^+; P(x) \rangle$ be the substructure of $\mathbf{B}$ whose elements are the positive real numbers. Let $\iota: \mathbf{A} \rightarrow \mathbf{B}: x \mapsto x$ be the inclusion map. With these choices (i) σ is a universal sentence, (ii) $\iota: \mathbf{A} \rightarrow \mathbf{B}$ is an embedding, (iii) ι does not preserve the satisfaction of σ because $\mathbf{A}$ satisfies σ but $\mathbf{B}$ does not.

For the second part of the problem, use the same embedding with a sentence τ equivalent to the negation of σ :

$$\tau := “(\exists x)(\neg P(x))”.$$

We have (i) τ is an existential sentence, (ii) $\iota: \mathbf{A} \rightarrow \mathbf{B}$ is an embedding, (iii) ι does not reflect the satisfaction of τ because $\mathbf{B}$ satisfies τ while $\mathbf{A}$ does not.