

Solutions to HW 6.

2. (Exercise 15 of Section 2.2.) Show that the addition relation, $\{\langle m, n, p \rangle \mid p = m + n\}$, is not definable in $(\mathbb{N}; \cdot)$. Suggestion: Consider an automorphism of $(\mathbb{N}; \cdot)$ that switches two primes.

The set $\{0\}$ is a definable singleton of $(\mathbb{N}; \cdot)$, so it must be fixed by every automorphism. However, there are multiplicative automorphisms of $\mathbb{N}$ that move other elements.

Each nonzero $m \in \mathbb{N}$ has a unique factorization into primes, hence has a unique factorization

$$m = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \dots,$$

which expresses the fact that a nonzero natural number m is uniquely determined by its exponent sequence $(e_1, e_2, e_3, \dots)$. This exponent sequence can be any sequence of natural numbers where all but finitely many of the coordinates is 0.

Following the hint, define an automorphism $\alpha: \mathbb{N} \rightarrow \mathbb{N}$ by swapping the role of the two primes, say 3 and 5. In more detail, define α by

- $\alpha(0) = 0$, and
- $\alpha(m) = n$ whenever $m = 2^{e_1} \cdot \underline{3^{e_2}} \cdot \underline{5^{e_3}} \cdot 7^{e_4} \dots$ and $n = 2^{e_1} \cdot \underline{5^{e_2}} \cdot \underline{3^{e_3}} \cdot 7^{e_4} \dots$.

The uniqueness of prime factorizations together with the fact that 0 is not a product of nonzero primes guarantee that this is a function. This function is its own inverse, so it is an permutation of $\mathbb{N}$. To verify that it is an automorphism, it is enough to check that $\alpha(m \cdot m') = \alpha(m) \cdot \alpha(m')$. For this, assume that $m = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot 7^{e_4} \dots$ and $m' = 2^{f_1} \cdot 3^{f_2} \cdot 5^{f_3} \cdot 7^{f_4} \dots$. Then

$$\begin{aligned} \alpha(m \cdot m') &= \alpha((2^{e_1} \cdot 3^{e_2} \dots) \cdot (2^{f_1} \cdot 3^{f_2} \dots)) \\ &= \alpha(2^{e_1+f_1} \cdot \underline{3^{e_2+f_2}} \cdot \underline{5^{e_3+f_3}} \dots) \\ &= 2^{e_1+f_1} \cdot \underline{5^{e_2+f_2}} \cdot \underline{3^{e_3+f_3}} \dots \\ &= (2^{e_1} \cdot 5^{e_2} \cdot 3^{e_3} \dots) \cdot (2^{f_1} \cdot 5^{f_2} \cdot 3^{f_3} \dots) \\ &= \alpha(m) \cdot \alpha(m'). \end{aligned}$$

One must also check that $\alpha(0 \cdot m) = \alpha(0) \cdot \alpha(m)$, $\alpha(m \cdot 0) = \alpha(m) \cdot \alpha(0)$, and $\alpha(0 \cdot 0) = \alpha(0) \cdot \alpha(0)$ for nonzero m , but these statements will hold because all of these values reduce to 0.¹

Now, we argue that α does not preserve addition. This will be enough to show that addition is not definable in $(\mathbb{N}; \cdot)$. That is, we need to show that $\alpha(m + n) = \alpha(m) + \alpha(n)$ does not ALWAYS hold.

$$\begin{aligned} \alpha(2 + 3) &= \alpha(5) \\ &= 3 \\ &\neq 7 \\ &= 2 + 5 \\ &= \alpha(2) + \alpha(3) \end{aligned}$$

Since $\alpha(2 + 3) \neq \alpha(2) + \alpha(3)$, the automorphism α does not preserve addition.

¹I believe that the author intends you to simply accept his claim that there is an automorphism that switches two primes, and does not require you to verify it. But, to do the problem, you have to know what kind of automorphism can switch two primes.

3. (Exercise 28(a) of Section 2.2.) For each of the following pairs of structures, show that they are not elementarily equivalent, by giving a sentence true in one and false in the other.

- (a) $(\mathbb{R}; \times)$ and $(\mathbb{R}^*; \times^*)$, where $\times$ is the usual multiplication operation on the real numbers, $\mathbb{R}^*$ is the set of non-zero reals, and $\times^*$ is $\times$ restricted to the non-zero reals.

The simplest sentence that separates these two structures might be “There exists a multiplicative absorbing element.” $(\mathbb{R}; \times)$ has one, namely 0, but $(\mathbb{R}^*; \times^*)$ does not. One may write this sentence formally as

$$(\exists x)(\forall y)((x \cdot y) = x).$$