

Solutions to HW 5.

2. (Exercises 1 Section 2.2)

- (a) $\Gamma \cup \{\alpha\} \models \varphi$ iff $\Gamma \models (\alpha \rightarrow \varphi)$.
- (b) $\varphi \models \psi$ iff $\models (\varphi \leftrightarrow \psi)$.

- (a) Both sides of the “iff” are equivalent to the following statement about valuations: For every structure $\mathbf{A}$ and every valuation v in $\mathbf{A}$, if v satisfies Γ and v satisfies α , then v satisfies φ .
- (b) $\varphi \models \psi$ is defined to mean the conjunction of $\varphi \models \psi$ and $\psi \models \varphi$. By Part (a), these two are each equivalent to $\models (\varphi \rightarrow \psi)$ and $\models (\psi \rightarrow \varphi)$, hence the conjunction of the two is equivalent to $\models (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. or $\models (\varphi \leftrightarrow \psi)$.

Alternatively, one could argue as follows: $\varphi \models \psi$ means that any valuation v in any structure $\mathbf{A}$ satisfies φ iff it satisfies ψ . This is equivalent to the statement that any valuation v in any structure $\mathbf{A}$ satisfies $(\varphi \leftrightarrow \psi)$.

3. (Exercise 2 of Section 2.2) Show that no one of the following sentences is logically implied by the other two.

- (a) $(\forall x)(\forall y)(\forall z)(P(x, y) \rightarrow (P(y, z) \rightarrow P(x, z)))$
- (b) $(\forall x)(\forall y)(P(x, y) \rightarrow (P(y, x) \rightarrow (x = y)))$
- (c) $((\forall x)(\exists y)P(x, y)) \rightarrow ((\exists y)(\forall x)P(x, y))$

There are many solutions. It may be helpful to discover solutions if we recognize that (a) is satisfied in a structure $\langle A; P(x, y) \rangle$ when the predicate P is transitive and (b) is satisfied in a structure $\langle A; P(x, y) \rangle$ when the predicate P is antisymmetric. Sentence (c) asserts that if every x is P -dominated by some y , then there is some y that P -dominates everything. The following structures show that two of the three sentences can be satisfied even when the third sentence fails. This is enough to show that no two of the sentences imply the third.

(a) $\wedge$ (b) $\wedge$ $\neg$ (c):

$\mathbf{A} = \langle \mathbb{N}; P(x, y) \rangle$ where $P(x, y)$ holds iff $x < y$. P is a total order, so (a) and (b) hold. Every x is $<$ -dominated by some y (e.g., $y = x + 1$) but there is no y that $<$ -dominates all x 's so (c) fails.

(a) $\wedge$ (c) $\wedge$ $\neg$ (b):

$\mathbf{A} = \langle \mathbb{N}; P(x, y) \rangle$ where $P(x, y)$ holds for every $x, y \in \mathbb{N}$. P is the universal binary relation, so (a) and (c) hold. But P is not antisymmetric ($P^{\mathbf{A}}(0, 1)$ and $P^{\mathbf{A}}(1, 0)$ both hold, but $0 \neq 1$), so (b) fails.

(b) $\wedge$ (c) $\wedge$ $\neg$ (a):

$\mathbf{A} = \langle \mathbb{N}; P(x, y) \rangle$ where $P(x, y)$ holds iff $x = S(y)$ where S is the successor function. P is antisymmetric since there is no pair (x, y) such that $x = S(y)$ and $y = S(x)$ holds. Sentence (c) holds, since $(\forall x)(\exists y)P(x, y)$ never holds. (For $x = 0$, there is no y such that $P(x, y)$ holds since $0 = S(y)$ has no solution $y \in \mathbb{N}$.) Sentence (a) (transitivity of P) fails, since $P^{\mathbf{A}}(2, 1)$ and $P^{\mathbf{A}}(1, 0)$ both hold, while $P(2, 0)$ does not hold.