

Solutions to HW 3.

2. (Exercise 3, Section 1.7) Show that from the corollary to the compactness theorem we can prove the compactness theorem itself (far more easily than we can starting from scratch).

Σ is **unsatisfiable** iff $\Sigma \models \perp$. Using the corollary in the case where $\tau := \perp$, there must exist a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \perp$. This shows that one consequence of the corollary is:

Σ is unsatisfiable iff there exists a finite subset $\Sigma_0 \subseteq \Sigma$ that is unsatisfiable.

If you negate both sides of this iff-statement you derive the Compactness Theorem:

Σ is satisfiable iff every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable.

3. Show that $\Gamma \models (\alpha \rightarrow \beta)$ holds if and only if $\Gamma \cup \{\alpha\} \models \beta$ holds. (Hint: Assume that one of them fails and that the failure is witnessed by a valuation v . Then show that v witnesses that the other fails.)

The following are equivalent for a valuation v :

- v satisfies Γ , but $v(\alpha \rightarrow \beta) = 0$.
- v satisfies Γ , but $v(\alpha) = 1$ and $v(\beta) = 0$.
- v satisfies $\Gamma \cup \{\alpha\}$, but $v(\beta) = 0$.

The existence of a v that satisfies the first item on this list is the definition of $\Gamma \not\models (\alpha \rightarrow \beta)$, while the existence of a v that satisfies the last item is the definition of $\Gamma \cup \{\alpha\} \not\models \beta$. Hence $\Gamma \not\models (\alpha \rightarrow \beta)$ is equivalent to $\Gamma \cup \{\alpha\} \not\models \beta$, and therefore $\Gamma \models (\alpha \rightarrow \beta)$ is equivalent to $\Gamma \cup \{\alpha\} \models \beta$.