

Solutions to HW 2.

2. (Exercise 2, Section 1.5) Show that NAND ($\uparrow$) and NOR ($\downarrow$) are the only binary connectives that are complete by themselves.

Let me rewrite truth tables for binary connectives like multiplication tables. For NAND and NOR we get

$\uparrow$	0	1
0	1	1
1	1	0

$\downarrow$	0	1
0	1	0
1	0	0

These two binary connectives are shown to be complete in the Example at the top of Page 52.

Let us argue that there is no other binary operation on $\{0, 1\}$ that is complete. Assume that $C(x, y)$ is a complete binary operation. Let's determine how to fill out its multiplication table:

C	0	1
0	a	b
1	c	d

$a, b, c, d \in \{0, 1\}$.

Claim 1. $C(0, 0) \neq 0$.

Reasoning. If $C(0, 0) = 0$, then C preserves the unary relation $\{0\}$. Therefore, it can only generate other operations that preserve this relation, and negation does not preserve this relation. Hence C could never generate negation.

Claim 2. $C(1, 1) \neq 1$.

Reasoning. Same argument as for Claim 1.

Claim 3. $C(0, 1) = C(1, 0)$.

Reasoning. Otherwise $C(0, 1) \neq C(1, 0)$ and we must have $(C(0, 1), C(1, 0)) = (1, 0)$ or $(C(0, 1), C(1, 0)) = (0, 1)$. In these case, C must be one of the operations

C	0	1
0	1	1
1	0	0

or

C	0	1
0	1	0
1	1	0

The first is $C(x, y) = \neg x$ and the second is $C(x, y) = \neg y$. These operations are fixed by conjugation by negation, so neither can be complete. In fact, these two operations generate the same Boolean operations on $\{0, 1\}$ as $\{\neg\}$, and this is not a complete set. This completes the argument that $C(0, 1) = C(1, 0)$.

The three claims, taken together, force the table of any complete binary connective to be

C	0	1
0	1	1
1	1	0

or

C	0	1
0	1	0
1	0	0

which means that $C(x, y) = x \uparrow y$ or $C(x, y) = x \downarrow y$.

3. (Exercise 5, Section 1.5) Show that $\{\top, \perp, \neg, \leftrightarrow, +\}$ is not complete.

Let me write 1 for $\top$ and 0 for $\perp$.

Claim. Any one of the connectives in $\{1, 0, \neg, +, \leftrightarrow\}$ may be constructed from the connectives in the subset $\{1, +\} \quad (\subseteq \{1, 0, \neg, \leftrightarrow, +\})$.

Proof of Claim. Write $F = G$ to mean F is constructed by G . Observe that on the rightmost side of each line we use only 1 and $+$ (and variables).

- (1) $1 = 1$.
- (2) $0 = 1 + 1$.
- (3) $\neg p = 1 + p$.
- (4) $p + q = p + q$
- (5) $p \leftrightarrow q = \neg(p + q) = p + q + 1$.

In this list, the Boolean operation on the left computes the same function as the Boolean operation on the right. $\square$

The Claim simplifies the problem to the following equivalent problem:

Show that $\{1, +\}$ is not complete.

To solve the simplified version, let's note that both 1 and $+$ commute with ternary addition modulo 2, but not all Boolean operations commute with ternary addition modulo 2 (e.g., $\wedge$ does not).

Verification that 1 and $+$ commute with ternary addition modulo 2: Let A be the graph of ternary addition modulo 2. That is, A consists of the set of 4-tuples of truth values (x, y, z, w) satisfying $x + y + z = w$.

- (1) (The constant function 1 preserves A) We must show that $(1, 1, 1, 1) \in A$. Equivalently, $1 + 1 + 1 = 1$.
 $\checkmark$
- (2) ($+$ preserves A) We must show that if $(x, y, z, w), (x', y', z', w') \in A$, then $(x+x', y+y', z+z', w+w') \in A$. If $x + y + z = w$ and $x' + y' + z' = w'$, then the fact that $+$ is commutative and associative implies that:

$$\begin{aligned} (x + x') + (y + y') + (z + z') &= (x + y + z) + (x' + y' + z') \quad \checkmark \\ &= w + w'. \end{aligned}$$

Verification that $\wedge$ does not commute with ternary addition modulo 2: $(1, 1, 0, 0), (0, 1, 1, 0) \in A$, but $(1, 1, 0, 0) \wedge (0, 1, 1, 0) = (0, 1, 0, 0) \notin A$. $\checkmark$

Since each operation in the set $\{1, +\}$ preserves A and $\wedge$ does not preserve A , $\{1, +\}$ is not complete. By the Claim, $\{1, 0, \neg, +, \leftrightarrow\}$ cannot be complete either. (More specifically, you cannot construct $\wedge$ from $\{1, 0, \neg, +, \leftrightarrow\}$.)