

The Kernel of a Function

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This concludes the explanation of why the equivalence relation defining properties correctly axiomatize kernels of functions.

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- ② (Equal parity) Define $m \sim n$ on $\mathbb{Z}$ if m and n have the same parity. (Both m and n are even or both are odd.) Then $\sim$ is an equivalence relation on $\mathbb{Z}$. The equivalence classes are $(0/\sim) = 2\mathbb{Z}$ and $(1/\sim) = 1 + 2\mathbb{Z}$.

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