

Galois Connections

Let S and T be sets (or classes), and let $R \subseteq S \times T$ be a relation. For each subset/subclass $U \subseteq S$ define

$$U^\perp = U^R := \text{class of all } t \in T \text{ such that } (\forall u \in U)((u, t) \in R)$$

and for each subclass $V \subseteq T$ define

$$V^\perp = {}^R V := \text{class of all } s \in S \text{ such that } (\forall v \in V)((s, v) \in R).$$

These are two mappings, $\perp: \mathcal{P}(S) \rightarrow \mathcal{P}(T)$ and $\perp: \mathcal{P}(T) \rightarrow \mathcal{P}(S)$. It is immediate from the definition of $\perp$ that:

Lemma 1. $U \subseteq V^\perp \iff U \times V \subseteq R \iff V \subseteq U^\perp$.

The bi-implication from leftmost to rightmost in Lemma 1 motivates the following definition.

Definition 2. A *Galois connection (GC)* between classes S and T is a pair of mappings $\perp: \mathcal{P}(S) \rightarrow \mathcal{P}(T)$ and $\perp: \mathcal{P}(T) \rightarrow \mathcal{P}(S)$ such that for all $U \subseteq S$ and $V \subseteq T$ it is the case that $U \subseteq V^\perp \iff V \subseteq U^\perp$.

Lemma 3. A Galois connection between S and T arises from a uniquely determined relation in the manner described before Lemma 1.

Proof. If a GC between S and T arises from a relation $R \subseteq S \times T$, then, according to Lemma 1, $(u, v) \in R$ iff $\{u\} \subseteq \{v\}^\perp$ iff $\{v\} \subseteq \{u\}^\perp$. So, given an arbitrary GC, the natural candidate for the associated relation is $R = \{(u, v) \in S \times T \mid v \in \{u\}^\perp\}$.

To see that the GC induced by this relation R is the same as the starting GC we must show that $U^R = U^\perp$ for all $U \subseteq S$. (Since everything is symmetric so far, the same argument will show ${}^R V = V^\perp$.) Note that R is defined so that the desired statement is true for singleton sets: $\{u\}^R = \{u\}^\perp$. So it suffices to prove that two GC's that agree on singletons are equal.

Claim 4. For any GC, $(\bigcup U_i)^\perp = \bigcap U_i^\perp$.

$$x \in (\bigcup U_i)^\perp \text{ iff } \bigcup U_i \subseteq \{x\}^\perp \text{ iff } \forall i (U_i \subseteq \{x\}^\perp) \text{ iff } \forall i (x \in U_i^\perp) \text{ iff } x \in \bigcap U_i^\perp.$$

$$\text{Now, for any } U \subseteq S \text{ we have } U^R = (\bigcup_{u \in U} \{u\})^R = \bigcap_{u \in U} \{u\}^R = \bigcap_{u \in U} \{u\}^\perp = U^\perp.$$

To prove the uniqueness of R , note that if $(u, v) \in R - R'$, then $v \in \{u\}^R - \{u\}^{R'}$, so different relations induce different GC's. $\square$

So a GC is just an alternative way to describe a binary relation.

Theorem 5. (*Properties of GC's*) Assume that $U \subseteq V \subseteq S$ and $X \subseteq Y \subseteq T$.

- (1) $\perp$ reverses inclusions: $U^\perp \supseteq V^\perp$ and $X^\perp \supseteq Y^\perp$.
- (2) $\perp\perp$ is increasing: $U \subseteq U^{\perp\perp}$ and $X \subseteq X^{\perp\perp}$.
- (3) $\perp\perp\perp = \perp$: $U^\perp = U^{\perp\perp\perp}$ and $X^\perp = X^{\perp\perp\perp}$.

(4) *The operations*

$$\text{cl} : \mathcal{P}(S) \rightarrow \mathcal{P}(S) : U \mapsto U^{\perp\perp}$$

and

$$\text{cl} : \mathcal{P}(T) \rightarrow \mathcal{P}(T) : X \mapsto X^{\perp\perp}$$

are closure operators.

(5) *A set is closed if and only if it is in the image of $\perp$.*

(6) *If $\mathcal{L}_S$ and $\mathcal{L}_T$ are the lattice of closed subsets of S and T , then $\perp : \mathcal{L}_S \rightarrow \mathcal{L}_T$ is a duality. (I.e., $\perp : \mathcal{L}_S^\partial \rightarrow \mathcal{L}_T$ is an isomorphism.)*

Proof. Item (1) follows from Claim 4 of Lemma 3.

For (2), it follows from the definition of GC that $U \subseteq U^{\perp\perp}$ iff $U^\perp \subseteq U^\perp$, which obviously holds.

For (3), apply $\perp$ to the inclusion in (2) and use (1) to get $U^\perp \supseteq U^{\perp\perp\perp}$. But by part (2) we have $U^\perp \subseteq (U^\perp)^{\perp\perp}$. Hence $U^\perp = U^{\perp\perp\perp}$.

For (4), we have that $\perp\perp$ is extensive from (2). To prove that $\perp\perp$ is isotone we use (1) twice:

$$U \subseteq V \implies U^\perp \supseteq V^\perp \implies U^{\perp\perp} \subseteq V^{\perp\perp}.$$

For idempotence, we $\perp$ the equation from (3) to get

$$U^{\perp\perp} = U^{\perp\perp\perp\perp} = (U^{\perp\perp})^{\perp\perp}.$$

For (5), note that any closed set is in the image of $\perp$, since $U = U^{\perp\perp}$ implies that U is the result of applying $\perp$ to $U^\perp$. Conversely, if $U = W^\perp$, then

$$U^{\perp\perp} = W^{\perp\perp\perp} = W^\perp = U,$$

so U is closed.

(6) follows from (5), (3), and (1). □

Examples.

(I) Let E be a finite dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$. Let $R \subseteq E \times E$ be the relation of orthogonality: $(u, v) \in R$ iff $\langle u, v \rangle = 0$. Then for $U \subseteq E$ the set $U^\perp$ is the orthogonal complement of U , and $U^{\perp\perp}$ is the subspace generated by U .

(II) The relation $\models$ induces a Galois Connection between the class of all L -structures and the class of all L -sentences.

(1) $\Gamma \models \alpha$ means the same as $\alpha \in \Gamma^{\perp\perp}$

(2) A closed class of L -sentences is called an *L-theory*.

(3) A closed class of L -structures is called an *elementary class*.

(4) If $\{\mathbf{A}\}^\perp = \{\mathbf{B}\}^\perp$, then $\mathbf{A}$ and $\mathbf{B}$ are called *elementarily equivalent*.