

Finite versus Infinite

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① $|X| \leq |Y|$ (or $|Y| \geq |X|$)

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Lemma.

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Exercise. Explain how Cantor's Theorem implies that $\mathcal{P}(\mathbb{N})$ is uncountable.