Expanding the Language of Set Theory

The Formal Language of Set Theory

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$$\begin{aligned}
0 &:= \emptyset \\
1 &:= S(0) \\
2 &:= S(1) \\
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Expanding the Language, 3

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Now we are in a position to introduce symbols for every natural number. We just need to write down the right formulas for them. (Try it!)

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The Power Set Axiom asserts that if X is a set, then the power set $\mathcal{P}(X)$ is also a set.

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