

Solutions to HW 7.

1. Determine whether the negation of the proposition P logically implies, is logically equivalent to, or is logically independent of proposition Q :

(i) $P = a \rightarrow b, Q = a \wedge (\neg b)$

The negation of P is logically equivalent to Q . We can verify this with truth tables.

a	b	$P = a \rightarrow b$	$\neg P = \neg(a \rightarrow b)$	$\neg b$	$Q = a \wedge (\neg b)$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

Note: This is an important example. If we want to justify the claim that an implication might be false (i.e., $P = a \rightarrow b$ does not hold for certain a and b), then we should explain why it is possible for the hypothesis “ a ” to be true and the conclusion “ b ” to be false (i.e., that $Q = a \wedge (\neg b)$ holds). This problem shows that establishing that $a \rightarrow b$ is false is equivalent to establishing that $a \wedge (\neg b)$ is true.

There is another way to do this problem using equivalences that we have already established.

$$\begin{aligned}
 \neg P &= \neg(a \rightarrow b) && \text{(From the Definition of } P\text{)} \\
 &\equiv \neg((\neg a) \vee b) && \text{ (“} \rightarrow \text{ is redundant”)} \\
 &\equiv (\neg(\neg a)) \wedge (\neg b) && \text{(De Morgan’s Law)} \\
 &\equiv a \wedge (\neg b) && \text{(Law of Double Negation)} \\
 &= Q && \text{(Definition of } Q\text{)}
 \end{aligned}$$

(ii) $P = (a \rightarrow b) \rightarrow a, Q = \neg a$.

The negation of P is logically equivalent to Q . We can verify this with truth tables.

a	b	$a \rightarrow b$	$P = (a \rightarrow b) \rightarrow a$	$\neg P$	$Q = \neg a$
0	0	1	0	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	1	0	0

(iii) $P = (a \rightarrow b) \wedge (b \rightarrow c), Q = (a \rightarrow c)$.

Let’s compare the truth tables of $\neg P$ and Q .

a	b	c	$X = a \rightarrow b$	$Y = b \rightarrow c$	$P = X \wedge Y$	$\neg P$	$Q = a \rightarrow c$
0	0	0	1	1	1	0	1
0	0	1	1	1	1	0	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	0	1
1	0	0	0	1	0	1	0
1	0	1	0	1	0	1	1
1	1	0	1	0	0	1	0
1	1	1	1	1	1	0	1

This is enough information to conclude that $\neg P$ and Q are logically independent. We see that it is possible for $\neg P$ to be true when Q is false (line 5 or line 7), so $\neg P$ does not logically imply Q . It is also possible for Q to be true when $\neg P$ is false (lines 1, 2, 4 or 8), so Q does not logically imply $\neg P$. (These facts can also be expressed by saying that $(\neg P) \rightarrow Q$ is not a tautology and $Q \rightarrow (\neg P)$ is not a tautology.)

Side comment: The table shows that P logically implies Q . This establishes the transitivity of implication. That is, $(a \rightarrow b) \wedge (b \rightarrow c)$ logically implies $(a \rightarrow c)$.

2. Write the following propositions in disjunctive normal form, assuming that each proposition is a function of p , q and r .

(i) $p \rightarrow r$

$$\underline{((\neg p) \wedge (\neg q) \wedge (\neg r))} \vee \underline{((\neg p) \wedge (\neg q) \wedge r)} \vee \underline{((\neg p) \wedge q \wedge (\neg r))} \vee \underline{((\neg p) \wedge q \wedge r)} \vee \underline{(p \wedge (\neg q) \wedge r)} \vee \underline{(p \wedge q \wedge r)}$$

(ii) $((p \rightarrow q) \rightarrow ((\neg p) \leftrightarrow r)).$

$$\underline{((\neg p) \wedge (\neg q) \wedge r)} \vee \underline{((\neg p) \wedge q \wedge r)} \vee \underline{(p \wedge (\neg q) \wedge (\neg r))} \vee \underline{(p \wedge (\neg q) \wedge r)} \vee \underline{(p \wedge q \wedge (\neg r))}$$

(iii) q

$$\underline{((\neg p) \wedge q \wedge (\neg r))} \vee \underline{((\neg p) \wedge q \wedge r)} \vee \underline{(p \wedge q \wedge (\neg r))} \vee \underline{(p \wedge q \wedge r)}$$

3. Write the following axioms of set theory as formal sentences.

(i) Extensionality.

$$(\forall A)(\forall B)((A = B) \leftrightarrow (\forall z)((z \in A) \leftrightarrow (z \in B)))$$

(ii) Pairing.

$$(\forall A)(\forall B)(\exists P)(\forall z)((z \in P) \leftrightarrow ((z = A) \vee (z = B)))$$

(iii) Power set.

$$(\forall A)(\exists P)(\forall z)((z \in P) \leftrightarrow \underline{(z \subseteq A)})$$

or

$$(\forall A)(\exists P)(\forall z)((z \in P) \leftrightarrow \underline{((\forall w)((w \in z) \rightarrow (w \in A)))})$$