

## Solutions to HW 6.

1. Show that  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}: (m, n) \mapsto 2^m(2n+1) - 1$  is a bijection.

( $f$  is injective) If  $f(m, n) = f(p, q)$ , then  $2^m(2n+1) - 1 = 2^p(2q+1) - 1$ , or  $2^m(2n+1) = 2^p(2q+1)$  (= a nonzero natural number). By the uniqueness of prime factorizations, every nonzero natural number is expressible in a unique way as a product of a power of 2 and an odd number, hence  $2^m = 2^p$  and  $2n+1 = 2q+1$ . From  $2^m = 2^p$  we derive  $m = p$  by unique factorization, and from  $2n+1 = 2q+1$  we derive  $2n = 2q$ , and then  $n = q$  with some arithmetic. This shows that  $f(m, n) = f(p, q)$  implies  $(m, n) = (p, q)$ .

( $f$  is surjective) If  $k \in \mathbb{N}$ , then we can solve  $2^x(2y+1) - 1 = k$  for  $x, y \in \mathbb{N}$ . Simply write this as  $2^x(2y+1) = k+1$ , then choose  $x$  so that  $2^x$  is the exact power of 2 that divides  $k+1$  and choose  $y$  so that  $2y+1$  is the odd number that remains after dividing  $k+1$  by  $2^x$ . The statement that  $2^x(2y+1) - 1 = k$  is solvable for  $x, y \in \mathbb{N}$  is exactly what it means for  $f$  to be surjective.

2. Show that if  $|X| = |Y|$ , then  $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$ .

I will write the solution in a general form below, but first let me explain the idea of the solution in small example. Let  $X = \{0, 1\}$  and  $Y = \{a, b\}$ . Let  $f: X \rightarrow Y$  be the bijection  $0 \mapsto a, 1 \mapsto b$ . Think of  $f$  as “renaming the elements 0, 1 using  $a, b$ ”. To prove that  $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$  we need to establish that there is a bijection  $\hat{f}: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ . Written out more fully, we need to establish that there is a bijection

$$\hat{f}: \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \rightarrow \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The idea is to take  $\hat{f}$  to be the function that “renames inside the braces”. That is  $\hat{f}(\emptyset) = \emptyset$ ,  $\hat{f}(\{0\}) = \{a\}$ ,  $\hat{f}(\{1\}) = \{b\}$ ,  $\hat{f}(\{0, 1\}) = \{a, b\}$ . So, the idea  $\hat{f}(\{0, 1\}) = \{f(0), f(1)\}$  may be expressed as “renaming a set with  $\hat{f}$  means renaming the elements inside the set with  $f$ ”.

Let’s write this down for general  $X$  and  $Y$ . Since  $|X| = |Y|$ , there is a bijection  $f: X \rightarrow Y$ . Let  $g: Y \rightarrow X$  be the inverse of  $f$ . Define functions  $\hat{f}: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  by the rule  $\hat{f}(S) = \{f(x) \in Y \mid x \in S\}$  and  $\hat{g}: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  by the rule  $\hat{g}(T) = \{g(y) \in X \mid y \in T\}$ . It is not hard to see that  $\hat{g}$  is the inverse of  $\hat{f}$ , as follows:

$$\hat{g} \circ \hat{f}(S) = \hat{g}(\{f(x) \in Y \mid x \in S\}) = \{g \circ f(x) \in X \mid x \in S\} = \{x \in X \mid x \in S\} = S$$

for any  $S \in \mathcal{P}(X)$ , and a similar argument shows that  $\hat{f} \circ \hat{g}(T) = T$  for any  $T \in \mathcal{P}(Y)$ .

Since  $\hat{f}: \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  is invertible, it is a bijection, hence  $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$ .

3. Let  $\text{Eq}(\mathbb{N})$  be the set of equivalence relations on  $\mathbb{N}$ . Show that  $|\mathcal{P}(\mathbb{N})| \leq |\text{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})|$ . Use Problem 2 to conclude that  $|\text{Eq}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$ .

Let  $\mathbb{N}^+$  be the set of nonzero natural numbers. According to the “Laws of Successor” (October 2 handout “arithmetic.pdf”), the successor function  $S: \mathbb{N} \rightarrow \mathbb{N}^+$  is a bijection, so  $|\mathbb{N}| = |\mathbb{N}^+|$ . By Problem 2,  $|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N}^+)|$ .

Now we define a function  $f: \mathcal{P}(\mathbb{N}^+) \rightarrow \text{Eq}(\mathbb{N})$  by defining  $f(S)$  to equal the equivalence relation on  $\mathbb{N}$  with one equivalence class equal to  $S \cup \{0\}$  and all other equivalence classes equal to singletons. That is, for  $S \in \mathcal{P}(\mathbb{N}^+)$ , define  $f(S) = (S \cup \{0\})^2 \cup \{(n, n) \mid n \in \mathbb{N}\}$ . The function  $f$  is injective, since if  $S, T \in \mathcal{P}(\mathbb{N}^+)$  and  $f(S) = f(T) = E$ , then the  $E$ -equivalence class containing 0 is  $S \cup \{0\} = T \cup \{0\}$ ; since both  $S$  and  $T$  consist of nonzero elements,  $S = T$ . Altogether this shows that

$$|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N}^+)| \leq |\text{Eq}(\mathbb{N})|. \quad (1)$$

Next, every equivalence relation on  $\mathbb{N}$  is a binary relation on  $\mathbb{N}$ , hence is a subset of  $\mathbb{N} \times \mathbb{N}$ . This means that  $\text{Eq}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$ , hence  $|\text{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})|$ . Since  $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ , we derive from Problem 2 that

$$|\text{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})| = |\mathcal{P}(\mathbb{N})|. \quad (2)$$

Combining (1) and (2) yields  $|\mathcal{P}(\mathbb{N})| \leq |\text{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N})|$ , so  $|\text{Eq}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$  by the Cantor-Bernstein-Schröder Theorem.