

Solutions to HW 4.

1. Exercise 2.5.4(7) parts (a) and (b): Let R_1 and R_2 be equivalence relations on a set X .
 - (a) Is it always the case that $R_1 \cap R_2$ is an equivalence relation?
 - (b) Is it always the case that $R_1 \cup R_2$ is an equivalence relation?

The answer to (a) is Yes, while the answer to (b) is No.

Explanation for Part (a):

- $R_1 \cap R_2$ is reflexive: Choose any $x \in X$. Since R_1 and R_2 are reflexive, $(x, x) \in R_1$ and $(x, x) \in R_2$. Hence $(x, x) \in R_1 \cap R_2$.
- $R_1 \cap R_2$ is symmetric: Choose any $(x, y) \in R_1 \cap R_2$. We must have $(x, y) \in R_1$ and $(x, y) \in R_2$, and these are symmetric relations, so $(y, x) \in R_1$ and $(y, x) \in R_2$. From this we get $(y, x) \in R_1 \cap R_2$, as desired.
- $R_1 \cap R_2$ is transitive: Choose any $(x, y), (y, z) \in R_1 \cap R_2$. We must have $(x, y), (y, z) \in R_1$ and $(x, y), (y, z) \in R_2$, and these are transitive relations, so $(x, z) \in R_1$ and $(x, z) \in R_2$. From this we get $(x, z) \in R_1 \cap R_2$, as desired.

Explanation for Part (b):

- Let $X = \{0, 1, 2\}$ and let
 $R_1 = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$ and
 $R_2 = \{(0, 0), (1, 1), (2, 2), (1, 2), (2, 1)\}$.
Both R_1 and R_2 are equivalence relations, but $R_1 \cup R_2$ is not because it is not transitive. The union $R_1 \cup R_2$ contains $(0, 1)$ and $(1, 2)$, but it does not contain $(0, 2)$.

2. For each of (a)-(c), give an example of a binary relation R on some set A where
 - (a) R is reflexive and symmetric, but not transitive.
 - (b) R is reflexive and transitive, but not symmetric.
 - (c) R is symmetric and transitive, but not reflexive.

There are many correct answers. Here are some:

- (a) Let $A = \mathbb{Z}$ be the set of integers, and let R be the relation consisting of pairs (x, y) where $|x - y| \leq 1$. This relation R is reflexive and symmetric, but $(0, 1), (1, 2) \in R$ and $(0, 2) \notin R$.
- (b) Let $A = \mathbb{Z}$ be the set of integers, and let R be the relation consisting of pairs (x, y) where $x \leq y$. This relation R is reflexive and transitive, but $(0, 1) \in R$ and $(1, 0) \notin R$.
- (c) Let $A = \mathbb{Z}$ be the set of integers, and let R be the empty binary relation on A . This relation R is symmetric and transitive, but $(0, 0) \notin R$.

3. 3. Explain why the following statement is true: A function is injective if and only if its kernel is the equality relation.

Assume that (i) $f: A \rightarrow B$ is injective and (ii) $(a, b) \in \ker(f)$. By (ii), f “makes a and b equal”, so $f(a) = f(b)$. By (i), $a = b$. This shows that $(a, b) \in \ker(f)$ implies $a = b$, so the kernel of f is the equality relation on A .

Next, assume that (i) $f: A \rightarrow B$ is a function and (ii) $\ker(f)$ is the equality relation. To show that f is injective, assume that $f(a) = f(b)$ for some $a, b \in A$. Then $(a, b) \in \ker(f)$, so by (ii) we have that $a = b$. This shows that $f(a) = f(b)$ implies $a = b$, so f is injective.