Solutions to HW 10.

1. (Exercise 4.8.3(g)) Prove the statement "No positive integers m and n exist such that $\frac{1}{m} + \frac{1}{n} > 2$."

Assume to the contrary that positive integers m and n exist satisfying $2 < \frac{1}{m} + \frac{1}{n}$. Since $m \geq 1$, we have

$$\frac{1}{m} = m^{-1} \leq 1^{-1} = 1.$$

A similar argument yields $\frac{1}{n} \leq 1$. Adding these inequalities yields $\frac{1}{m} + \frac{1}{n} \leq 2$. With our original assumption this yields

$$2 < \frac{1}{m} + \frac{1}{n} \leq 2,$$

or $2 < 2$, which is false. This contradiction proves the result.

2. (Exercise 4.8.3(h).) Prove the statement "The number $\sqrt[3]{3}$ is irrational."

Assume to the contrary that $\sqrt[3]{3}$ is rational. We can then write the positive real number $\sqrt[3]{3}$ in the form $\frac{m}{n}$ where m and n are positive integers satisfying $\gcd(m, n) = 1$. Rewriting $\sqrt[3]{3} = \frac{m}{n}$ as $\sqrt[3]{3}n = m$ and cubing both sides leads to $3n^3 = m^3$. This equality implies that m^3 is divisible by 3, so m must be divisible by 3 since 3 is prime. Write $m = 3k$ for an appropriate positive integer k. We now have $3n^3 = (3k)^3 = 27k^3$, so $n^3 = 9k^3$, so n^3 is divisible by 3, so n is divisible by 3. So far we have shown that both m and n are divisible by 3, which contradicts the assumption that $\gcd(m, n) = 1$.

3. (Exercise 4.8.3(i).) Prove the statement "If $x + y$ is an irrational number, then x is an irrational number or y is an irrational number."

The contrapositive form of this statement is: "If x and y are rational, then $x + y$ is rational." To show this, write $x = \frac{a}{b}$ and $y = \frac{c}{d}$ where $a, b, c, d \in \mathbb{Z}$ and $b \neq 0 \neq d$. Now,

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

where $ad + bc, bd \in \mathbb{Z}$ and $bd \neq 0$. This is enough to show that $x + y$ is rational.