

Practice with Deductions!

- (1) (Exercise 4 of Section 2.4.) Give a deduction of

$$(\forall x)\phi \rightarrow (\exists x)\phi.$$

(Note that you should not merely prove that such a deduction exists. You are instead asked to write out the entire deduction.)

Hints!

- (a) Recall that Enderton avoids the symbol $\exists$ in his deductive system. Therefore, read “ $(\exists x)\phi$ ” in the problem statement as an abbreviation for “ $\neg(\forall x)\neg\phi$ ”.

- (b) One way to solve this problem involves showing that the following is a tautology:

$$(\star) \quad ((\forall x)\phi \rightarrow \phi) \rightarrow \left(((\forall x)(\neg\phi) \rightarrow (\neg\phi)) \rightarrow ((\forall x)\phi \rightarrow (\neg(\forall x)(\neg\phi))) \right).$$

This formula has the structure $A \rightarrow (B \rightarrow C)$ for appropriate A, B, C .

Try to show that $(\star)$ is a tautology.

Suppose we know that $(\star)$ is a tautology. Since $(\star)$ has the structure $A \rightarrow (B \rightarrow C)$, then a proof exists of the following form:

$$A, \quad B, \quad A \rightarrow (B \rightarrow C), \quad B \rightarrow C, \quad C.$$

If we want to include explanations, we could write it in the form:

A	Axiom of type 2
B	Axiom of type 2
$A \rightarrow (B \rightarrow C)$	Axiom of type 1 (=tautology)
$B \rightarrow C$	MP applied to lines 1 and 3
C	MP applied to lines 2 and 4. $\square$

But these are abbreviations. The actual proof is:

$(\forall x)\phi \rightarrow \phi$	Axiom of type 2
$(\forall x)(\neg\phi) \rightarrow (\neg\phi)$	Axiom of type 2
$((\forall x)\phi \rightarrow \phi) \rightarrow \left(((\forall x)(\neg\phi) \rightarrow (\neg\phi)) \rightarrow ((\forall x)\phi \rightarrow (\neg(\forall x)(\neg\phi))) \right)$	Axiom of type 1
$\left(((\forall x)(\neg\phi) \rightarrow (\neg\phi)) \rightarrow ((\forall x)\phi \rightarrow (\neg(\forall x)(\neg\phi))) \right)$	MP applied to lines 1 and 3
$(\forall x)\phi \rightarrow (\neg(\forall x)(\neg\phi))$	MP applied to lines 2 and 4. $\square$

Now, to show that $(\star)$ is a tautology, show that it is a substitution instance of

$$(\star\star) \quad (p \rightarrow q) \rightarrow ((r \rightarrow \neg q) \rightarrow (p \rightarrow \neg r)),$$

and that $(\star\star)$ is a propositional tautology.