The Completeness Theorem, Part 1

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Corollary 24G. Assume that $\Gamma \vdash \varphi^x_c$ where c is a constant that does not occur in $\Gamma \cup \{\varphi\}$. There is a Γ-proof of $(\forall x)\varphi$ in which c does not occur.

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$$\Gamma \vdash \neg(\forall x)\varphi \quad \text{and} \quad \Gamma \vdash \varphi^x_c.$$

Preparing Γ, 4 (continued)

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Last line of previous slide:

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By Corollary 24G, the fact that c does not occur in Γ implies that from the second of these we get:

Preparing Γ, 4 (continued)

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Preparing Γ, 4 (continued)

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which is a contradiction to the consistency of Γ. $\square$

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For uncountable languages, take unions at limits.

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For uncountable languages, take unions at limits. Forming a union of all L_k's and Γ_k's, we get a consistent extension $\Gamma' \ (\supseteq \Gamma)$ in $L' \ (\supseteq L)$ where $\neg(\forall x_k)\varphi \to \neg\varphi^{x_k}_{c_k} \in \Gamma'$ for all k.

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$$(\exists x_k)\varphi(x_k) \to \varphi(c_k) \in \Gamma'.$$

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