

The Compactness Theorem for Propositional Logic

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The Compactness Theorem explains how to go from “Finitely Complete” to “Complete”.

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Let's refer to this by saying that $\models$ is 'monotone'.

Next, observe that $\vdash$ has the following 'compactness' property:

If $\Gamma \vdash \alpha$, then there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \alpha$.

Since we want $\vdash$ to capture $\models$, we would like the same 'compactness' property to hold for $\models$:

If $\Gamma \models \alpha$, then there exists a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \alpha$.

The 'Compactness Theorem for Propositional Logic' asserts that the last displayed line is a true statement.

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