

Rules of Arithmetic on $\mathbb{N}$

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Exponentiation

$$m^0 \quad := 1 \quad (\text{IC})$$

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If both steps are accomplished, you have shown that S_n is true for all n .

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$$m + (n + S(k)) = m + S(n + k) \quad ((\text{RR}), +)$$

The Associative Law: $m + (n + k) = (m + n) + k$

This will be proved by induction, but should we argue by induction on m ? n ? or k ?

Experience shows that, since addition is defined by recursion on its last variable, we should prove properties of addition by induction on the last variable, k . Thus, for fixed values of m and n , the k -th statement to be proved is

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This proves that $0 + m = m$ for all $m \in \mathbb{N}$.

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At this point we should expect to prove the Inductive Step.

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Proving the Lemma

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Lemma. $m + 1 = 1 + m$ holds for all $m \in \mathbb{N}$.

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Completing the proof of the Commutative Law

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$$m + S(n) = S(m + n) \qquad ((\text{RR}), +)$$

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$$\begin{aligned} m + S(n) &= S(m + n) && ((\text{RR}), +) \\ &= S(n + m) \end{aligned}$$

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$$\begin{aligned} m + S(n) &= S(m + n) && ((\text{RR}), +) \\ &= S(n + m) && (\text{IH}) \end{aligned}$$

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This proves that $m + n = n + m$ for all $m, n \in \mathbb{N}$.

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Proof. If $n \neq 0$, then $n = S(k)$ by Part (a) of the Laws of Successor.

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Proof. If $n \neq 0$, then $n = S(k)$ by Part (a) of the Laws of Successor. Then $0 = m + n = m + S(k) = S(m + k)$, contradicting that 0 is not a successor. Hence $0 = m + n$ forces $n = 0$.

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$$\begin{aligned} m &= m + 0 \\ &= n + 0 \end{aligned} \quad ((\text{IC}), +)$$

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Assume that $m + S(k) = n + S(k)$. Then by $((\text{RR}), +)$ we have $S(m + k) = S(n + k)$. But the successor function is injective, by Part (b) of the Laws of Successor.

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Assume that $m + S(k) = n + S(k)$. Then by $((\text{RR}), +)$ we have $S(m + k) = S(n + k)$. But the successor function is injective, by Part (b) of the Laws of Successor. Thus, $m + k = n + k$.

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Since we have already proved the Commutative Law, the Left Cancellation Law is also valid: $k + m = k + n$ implies $m = n$.

Right Cancellation: $m + k = n + k$ implies $m = n$.

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Since we have already proved the Commutative Law, the Left Cancellation Law is also valid: $k + m = k + n$ implies $m = n$.

(Proof: $k + m = k + n$ implies $m + k = n + k$ implies $m = n$.)