

Definitions and Laws of Arithmetic on $\mathbb{N}$.

Abbreviations

$$S(x) := x \cup \{x\}, \quad 0 := \emptyset, \quad 1 := S(0), \quad 2 := S(1), \quad 3 := S(2), \quad \text{ETC}$$

Addition

$$\begin{aligned} m + 0 &:= m & (\text{IC}) \\ m + S(n) &:= S(m + n) & (\text{RR}) \end{aligned}$$

Multiplication

$$\begin{aligned} m \cdot 0 &:= 0 & (\text{IC}) \\ m \cdot S(n) &:= m \cdot n + m & (\text{RR}) \end{aligned}$$

Exponentiation

$$\begin{aligned} m^0 &:= 1 & (\text{IC}) \\ m^{S(n)} &:= m^n \cdot m & (\text{RR}) \end{aligned}$$

Laws of successor. (These should be proved first.)

- (a) 0 is not a successor. Every nonzero natural number is a successor.
- (b) Successor is injective. ($S(m) = S(n)$ implies $m = n$.)

Laws of addition. (Provable by induction or from the definitions.)

- (a) $S(n) = n + 1$, $1 + n = S(n)$
- (b) (Associative Law) $m + (n + k) = (m + n) + k$
- (c) (Unit Law for 0) $m + 0 = 0 + m = m$
- (d) (Commutative Law) $m + n = n + m$
- (e) (Irreducibility of 0) $m + n = 0$ implies $m = n = 0$.
- (f) (Cancellation) $m + k = n + k$ implies $m = n$.

Laws of multiplication (and addition).

- (a) (Associative Law) $m \cdot (n \cdot k) = (m \cdot n) \cdot k$
- (b) (Unit Law for 1) $m \cdot 1 = 1 \cdot m = m$
- (c) (Commutative Law) $m \cdot n = n \cdot m$
- (d) (0 is absorbing) $m \cdot 0 = 0 \cdot m = 0$
- (e) (No Proper Zero Divisors Property) If $m \cdot n = 0$, then $m = 0$ or $n = 0$.
- (f) (Irreducibility of 1) $m \cdot n = 1$ implies $m = n = 1$
- (g) (Distributive Law) $m \cdot (n + k) = (m \cdot n) + (m \cdot k)$

Laws of exponentiation (and multiplication and addition).

- (a) $m^0 = 1$, $m^1 = m$, $0^m = 0$ (if $m > 0$), and $1^m = 1$.
- (b) $m^{n+k} = m^n \cdot m^k$
- (c) $(m \cdot n)^k = m^k \cdot n^k$
- (d) $(m^n)^k = m^{n \cdot k}$