The Reflection Theorem

Preliminaries about absoluteness

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holds. The last existential quantifier asserts that, for some $\bar{y} \in M$ and some φ_j, there is some $x \in N$ that witnesses the failure of $\varphi_j^N(x, \bar{y})$, even though the preceding clause of the sentence expresses that there is no such $x \in M$. Let's call such an x $(= x_{\bar{y}} \in N - M)$ a 'witness' to nonabsoluteness for M, N.

Guaranteeing absoluteness

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Theorem 15.45, NST.

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Corollary 15.6, NST.

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$$\mathrm{ZF} \vdash (\forall\alpha)(\exists\beta > \alpha)[\varphi_0, \ldots, \varphi_n \text{ are absolute for } V_\beta, V.]$$

A refinement

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Theorem 15.7, NST.

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Guaranteeing transitivity

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Cumulative result

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This implies that if ZFC has a model, then there is a model V of ZFC that contains a <u>set</u> M that is a countable, transitive model of ZFC. (The bijection between M and ω that establishes the countability of M belongs to V, not M.)