Propositional logic

Propositions

A **proposition** is a declarative statement.
It may be true or false.
A **propositional variable** is a symbol for a proposition.

Examples.

- A = "Alice is a genius". The symbol "A" is a propositional variable. It denotes the proposition "Alice is genius". "A" may assume the truth value T or F.
- B = "Bob is a genius".
- R = "It is raining".
- W = "The ground is wet".

Compound propositions

We can express complex propositions in terms of simpler ones using the **logical connectives**: and ($\wedge$, conjunction), or ($\vee$, disjunction), not ($\neg$, negation), if-then ($\rightarrow$, implication), if-and-only-if ($\leftrightarrow$, bi-implication).

R	W	$R \wedge W$
0	0	0
0	1	0
1	0	0
1	1	1

R	W	$R \vee W$
0	0	0
0	1	1
1	0	1
1	1	1

R	$\neg R$
0	1
1	0

R	W	$R \rightarrow W$
0	0	1
0	1	1
1	0	0
1	1	1

R	W	$R \leftrightarrow W$
0	0	1
0	1	0
1	0	0
1	1	1

R	W	$R \oplus W = R \veebar W$
0	0	0
0	1	1
1	0	1
1	1	0

Example: a truth table for a compound proposition

Write the truth table for $P := R \vee (((A \wedge B) \rightarrow (\neg R)))$.

A	B	R	$A \wedge B$	$\neg R$	$((A \wedge B) \rightarrow (\neg R))$	P
0	0	0	0	1	1	1
0	0	1	0	0	1	1
0	1	0	0	1	1	1
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	0	1	1
1	1	0	1	1	1	1
1	1	1	1	0	0	1

This proposition P is a **tautology**, because it assumes the value “true” under any truth assignment to the propositional variables. This means that P is true because of its logical structure alone, and not because of the truth values of its variables.

Tautologies, contradictions, logical equivalence

A (compound) proposition is a **tautology** it assumes the value "true" under any truth assignment to the propositional variables.

A (compound) proposition is a **contradiction** it assumes the value "false" under any truth assignment to the propositional variables.

Two propositions are **(logically) equivalent** if they assume the same truth value under any truth assignment to the propositional variables.
(Write $P \equiv Q$.)

Facts and examples

Facts.

- The negation of a tautology is a contradiction, and vice versa.
- $P \equiv Q$ iff $P \leftrightarrow Q$ is a tautology.
- Logical equivalence is an equivalence relation on the set of all propositions in a given set of variables.

Examples.
Some Tautologies: $(P \vee (\neg P))$, $(P \to P)$, $((P \wedge Q) \to P)$.
Some Equivalences:

- $\neg(\neg P) \equiv P$,
- $P \to Q \equiv (\neg Q) \to (\neg P)$,
- (De Morgan's Laws) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$, and $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
- ($\veebar$ is redundant) $(P \veebar Q) \equiv \neg(P \leftrightarrow Q)$
- ($\leftrightarrow$ is redundant) $(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P)$
- ($\to$ is redundant) $(P \to Q) \equiv (\neg P) \vee Q$

Logical implication, logical equivalence, logical independence

If $P \to Q$ is a tautology, we say that P **logically implies** Q. P is **logically equivalent** to Q (denoted $P \equiv Q$) if any of the following hold:

- P logically implies Q and Q logically implies P.
- Both $P \to Q$ and $Q \to P$ are tautologies.
- $P \leftrightarrow Q$ is a tautology.
- P and Q have the same truth table.

P and Q fail to be logically equivalent when at least one of the implications $P \to Q$ or $Q \to P$ fails to be a tautology. If *both* $P \to Q$ and $Q \to P$ fail to be tautologies, then we say that P and Q are **logically independent**.

Examples.

- $H \to C$ is logically equivalent to $(\neg C) \to (\neg H)$.
- $H \to C$ is logically independent of $C \to H$.
- $H \to C$ is logically independent of $(\neg H) \to (\neg C)$.

Direct implication, contrapositive, converse, inverse

Give the (direct) implication $H \to C$, we call $(\neg C) \to (\neg H)$ the **contrapositive (implication)**, $C \to H$ the **converse (implication)**, and $(\neg H) \to (\neg C)$ the **inverse (implication)**,

H	C	$\neg H$	$\neg C$	$H \to C$	$(\neg C) \to (\neg H)$	$C \to H$	$(\neg H) \to (\neg C)$
0	0	1	1	1	1	1	1
0	1	1	0	1	1	0	0
1	0	0	1	0	0	1	1
1	1	0	0	1	1	1	1
				direct	contrapositive	converse	inverse

Note that the inverse is the contrapositive of the converse, hence it is logically independent of the original implication.

Example. Compare an example statement to its inverse:

"If $\underbrace{\text{we adopt the new policy}}_{H}$, then $\underbrace{\text{things will get better}}_{C}$."

"If $\underbrace{\text{we do not adopt the new policy}}_{\neg H}$, then $\underbrace{\text{things will not get better}}_{\neg C}$."

Disjunctive normal form, I

A **monomial** in the variables $\{A, B, C, D\}$ is a $\wedge$ (= conjunction) of $\pm$ variables:

$$(\neg A) \wedge B \wedge C \wedge (\neg D).$$

The truth table of a monomial has exactly one row whose value is $T = 1$:

A	B	C	D	$(\neg A) \wedge B \wedge C \wedge (\neg D)$
0	0	0	0	0
				$\vdots$
0	1	1	0	1
				$\vdots$
1	1	1	1	0

The monomial $(\neg A) \wedge B \wedge C \wedge (\neg D)$ assumes value 1 iff $A = 0, B = 1, C = 1, D = 0$.

Disjunctive normal form, II

If M is a monomial that has 1 only in row i, and N is a monomial that has 1 only in row j, then $M \vee N$ has 1 only in rows i and j. Using this idea, one can create a proposition with any prescribed truth table of the form "$\bigvee$ monomials", a disjunction (= $\vee$) of monomials. This form is called Disjunctive Normal Form (DNF).

Small example. Create a proposition with truth table

A	B	C	?
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$((\neg A) \wedge B \wedge (\neg C)) \vee (A \wedge (\neg B) \wedge C) \vee (A \wedge B \wedge C)$$

Disjunctive normal form, III

Using the procedure just described, it is easy to see why the following is true:

Theorem. Every propositional formula is logically equivalent to a formula in DNF = $\bigvee(\wedge \pm \text{variables})$.

Corollary. The symbols $\wedge, \vee, \neg$ are a "complete" set of logical connectives, in the sense that any proposition is logically equivalent to one expressed with $\{\wedge, \vee, \neg\}$ + propositional variables.

DNF depends on the choice of variables

If the set of propositional variables to be considered is $\{A\}$, then the DNF for proposition A is just $A = A$. But if the set of propositional variables to be considered is $\{A, B\}$, then the DNF for A is $(A \wedge (\neg B)) \vee (A \wedge B)$, since

A	B	A
0	0	0
0	1	0
1	0	1
1	1	1

If the set of propositional variables to be considered is $\{A, B, C\}$, then the DNF for A is

$$(A \wedge (\neg B) \wedge (\neg C)) \vee (A \wedge (\neg B) \wedge C) \vee (A \wedge B \wedge (\neg C)) \vee (A \wedge B \wedge C)$$

Exercise!

Write $p \to r$ in disjunctive normal form in the variables p, q, r.

Solution: (i) create the truth table, (ii) find the monomials, (iii) write answer.

p	q	r	$p \to r$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	**0**
1	0	1	1
1	1	0	**0**
1	1	1	1

$$((\neg p) \wedge (\neg q) \wedge (\neg r)) \;\vee\; ((\neg p) \wedge (\neg q) \wedge r) \;\vee\; ((\neg p) \wedge q \wedge (\neg r))$$
$$\vee\; ((\neg p) \wedge q \wedge r) \;\vee\; (p \wedge (\neg q) \wedge r) \;\vee\; (p \wedge q \wedge r)$$