7. Ordinals, I

In this chapter we introduce the ordinals and give basic facts about them.

A set A is *transitive* iff $\forall x \in A \forall y \in x(y \in A)$; in other words, iff every element of A is a subset of A. This is a very important notion in the foundations of set theory, and it is essential in our definition of ordinals. An *ordinal number*, or simply an *ordinal*, is a transitive set of transitive sets. We use the first few Greek letters to denote ordinals. If α, β, γ are ordinals and $\alpha \in \beta \in \gamma$, then $\alpha \in \gamma$ since γ is transitive. This partially justifies writing $\alpha < \beta$ instead of $\alpha \in \beta$ when α and β are ordinals. This helps the intuition in thinking of the ordinals as kinds of numbers. We also define $\alpha \leq \beta$ iff $\alpha < \beta$ or $\alpha = \beta$.

By a vacuous implication we have:

Proposition 7.1. *$\emptyset$ is an ordinal.* □

Because of this proposition, the empty set is a number; it will turn out to be the first nonnegative integer, the first ordinal, and the first cardinal number. For this reason, we will use 0 and $\emptyset$ interchangably, trying to use 0 when numbers are involved, and $\emptyset$ when they are not.

Proposition 7.2. *If α is an ordinal, then so is $\alpha \cup \{\alpha\}$.*

Proof. If $x \in y \in \alpha \cup \{\alpha\}$, then $x \in y \in \alpha$ or $x \in y = \alpha$. Since α is transitive, $x \in \alpha$ in either case. So $\alpha \cup \{\alpha\}$ is transitive. Clearly every member of $\alpha \cup \{\alpha\}$ is transitive. □

We denote $\alpha \cup \{\alpha\}$ by $\alpha +' 1$. After introducing addition of ordinals, it will turn out that $\alpha + 1 = \alpha +' 1$ for every ordinal α, so that the prime can be dropped. This ordinal $\alpha +' 1$ is the *successor* of α. We define $1 = 0 +' 1$, $2 = 1 +' 1$, etc. (up through 9; no further since we do not want to try to justify decimal notation).

Proposition 7.3. *If A is a set of ordinals, then $\bigcup A$ is an ordinal.*

Proof. Suppose that $x \in y \in \bigcup A$. Choose $z \in A$ such that $y \in z$. Then z is an ordinal, and $x \in y \in z$, so $x \in z$; hence $x \in \bigcup A$. Thus $\bigcup A$ is transitive.

If $u \in \bigcup A$, choose $v \in A$ such that $u \in v$. then v is an ordinal, so u is transitive. □

We sometimes write $\sup(A)$ for $\bigcup A$. In fact, $\bigcup A$ is the least ordinal $\geq$ each member of A. We prove this shortly.

Proposition 7.4. *Every member of an ordinal is an ordinal.*

Proof. Let α be an ordinal, and let $x \in \alpha$. Then x is transitive since all members of α are transitive. Suppose that $y \in x$. Then $y \in \alpha$ since α is transitive. So y is transitive, since all members of α are transitive. □

Theorem 7.5. *$\forall x(x \notin x)$.*

Proof. Suppose that x is a set such that $x \in x$. Let $y = \{x\}$. By the foundation axiom, choose $z \in y$ such that $z \cap y = \emptyset$. But $z = x$, so $x \in z \cap y$, contradiction. □

Theorem 7.6. *There does not exist a set which has every ordinal as a member.*

Proof. Suppose to the contrary that A is such a set. Let $B = \{x \in A : x \text{ is an ordinal}\}$. Then B is a set of transitive sets and B itself is transitive. Hence B is an ordinal. So $B \in A$. It follows that $B \in B$. contradicting Theorem 7.5. □

Theorem 7.6 is what happens in our axiomatic framework to the Burali-Forti paradox.

Theorem 7.7. *If α and β are ordinals, then $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$.*

Proof. Suppose that this is not true, and let α and β be ordinals such that $\alpha \neq \beta$, $\alpha \notin \beta$, and $\beta \notin \alpha$. Let $A = (\alpha +' 1) \cup (\beta +' 1)$. Define $B = \{\gamma \in A : \exists \delta \in A[\gamma \neq \delta, \gamma \notin \delta, \text{ and } \delta \notin \gamma]\}$. Thus $\alpha \in B$, since we can take $\delta = \beta$. So $B \neq \emptyset$. By the foundation axiom, choose $\gamma \in B$ such that $\gamma \cap B = \emptyset$. Let $C = \{\delta \in A : \gamma \neq \delta, \gamma \notin \delta, \text{ and } \delta \notin \gamma\}$. So $C \neq \emptyset$ since $\gamma \in B$. By the foundation axiom choose $\delta \in C$ such that $\delta \cap C = \emptyset$. We will now show that $\gamma = \delta$, which is a contradiction.

Suppose that $\varepsilon \in \gamma$. Then $\varepsilon \notin B$. Clearly $\varepsilon \in A$, so it follows that $\forall \varphi \in A[\varepsilon = \varphi$ or $\varepsilon \in \varphi$ or $\varphi \in \varepsilon]$. Since $\delta \in A$ we thus have $\varepsilon = \delta$ or $\varepsilon \in \delta$ or $\delta \in \varepsilon$. If $\varepsilon = \delta$ then $\delta \in \gamma$, contradiction. If $\delta \in \varepsilon$, then $\delta \in \gamma$ since γ is transitive, contradiction. So $\varepsilon \in \delta$. This proves that $\gamma \subseteq \delta$.

Suppose that $\varepsilon \in \delta$. Then $\varepsilon \notin C$. It follows that $\gamma = \varepsilon$ or $\gamma \in \varepsilon$ or $\varepsilon \in \gamma$. If $\gamma = \varepsilon$ then $\gamma \in \delta$, contradiction. If $\gamma \in \varepsilon$ then $\gamma \in \delta$ since δ is transitive, contradiction. So $\varepsilon \in \gamma$. This proves that $\delta \subseteq \gamma$.

Hence $\delta = \gamma$, contradiction. □

Proposition 7.8. *$\alpha \leq \beta$ iff $\alpha \subseteq \beta$.*

Proof. $\Rightarrow$: Assume that $\alpha \leq \beta$ and $x \in \alpha$. Then $x < \alpha \leq \beta$, so $x < \beta$ since β is transitive. Hence $x \in \beta$. Thus $\alpha \subseteq \beta$.

$\Leftarrow$: Assume that $\alpha \subseteq \beta$. If $\beta < \alpha$, then $\beta < \beta$, hence $\beta \in \beta$, contradicting Theorem 7.5. Hence $\alpha \leq \beta$ by Theorem 7.7. □

Proposition 7.9. *$\alpha < \beta$ iff $\alpha \subset \beta$.*

Proof. $\alpha < \beta$ iff ($\alpha \leq \beta$ and $\alpha \neq \beta$) iff ($\alpha \subseteq \beta$ and $\alpha \neq \beta$) (by Proposition 7.8) iff $\alpha \subset \beta$. □

Proposition 7.10. *$\alpha < \beta$ iff $\alpha +' 1 \leq \beta$.*

Proof. $\Rightarrow$: Assume that $\alpha < \beta$. If $\beta < \alpha +' 1$, then $\beta \in \alpha \cup \{\alpha\}$, so $\beta \in \alpha$ or $\beta = \alpha$. Since $\alpha \in \beta$, this implies that $\beta \in \beta$, contradicting Theorem 7.5. Hence by Theorem 7.7, $\alpha +' 1 \leq \beta$.

$\Leftarrow$: Assume that $\alpha +' 1 \leq \beta$. Then $\alpha < \alpha +' 1 \leq \beta$, so $\alpha < \beta$. □

Proposition 7.11. *There do not exist ordinals α, β such that $\alpha < \beta < \alpha +' 1$.* □

Theorem 7.12. *If A is a set of ordinals, then $\alpha \leq \bigcup A$ for each $\alpha \in A$, and if β is an ordinal such that $\alpha \leq \beta$ for all $\alpha \in A$ then $\bigcup A \leq \beta$.*