Logic Exercises. (+ some solution sketches!)

(1) Write the following formulas formally in the appropriate language.

(a) (Use the language of equality)
κ_3: "There exist three distinct elements."

$$(\exists x_0)(\exists x_1)(\exists x_2)(\neg(x_0 = x_1) \wedge \neg(x_1 = x_2) \wedge \neg(x_0 = x_2))$$

(b) (Use the language of set theory.)
$\varphi_{\subseteq}(x, y)$: "$x \subseteq y$".

$$(\forall z)((z \in x) \to (z \in y))$$

(c) (Use the language of group theory.)
ε: "Any two multiplicative identity elements are equal".

Suppose that $\varphi_{\text{id}}(x)$ expresses that x is a multiplicative identity element. The desired sentence may written with this abbreviation as

$$(\forall x)(\forall y)((\varphi_{\text{id}}(x) \wedge \varphi_{\text{id}}(y)) \to (x = y))$$

One way to write $\varphi_{\text{id}}(x)$ is as "$(\forall z)((x \cdot z = z) \wedge (z \cdot x = z))$". Subsituting this for the abbreviation yields:

$$(\forall x)(\forall y)(((\forall z)((x \cdot z = z) \wedge (z \cdot x = z)) \wedge (\forall z)((y \cdot z = z) \wedge (z \cdot y = z))) \to (x = y))$$

The above sentence is not in prenex form. In general, a sentence of the form

$$(\forall x)\ ((\forall z)P(x, z)) \to Q(x)),$$

where $Q(x)$ does not depend on z, has prenex form

$$(\forall x)(\exists z)(P(x, z) \to Q(x)).$$

(2) Put the following in prenex form:

$$((\forall x)(x = x) \leftrightarrow (\exists x)(x = x))$$

$$(\exists y)(\exists z)(\forall w)(\forall x)(((y = y) \to (z = z)) \wedge ((w = w) \to (x = x)))$$

(This puts the quantifiers in front. More work is required if you want the quantifier-free part to be written in DNF.)