The Forcing Theorem

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- When |X| = 1, RO(\mathbb{X}) is the 2-element BA.

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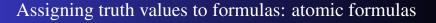
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