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(Proof may be found in NST, pages 606-609.)

Two Lemmas about $\mathrm{RO}(\mathbb{P})$

Lemma A.

Lemma A. (NST, Theorem 27.2(i))

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1. Show $U^{\text{open}} \cap V^{\text{open}} = \emptyset$ implies $U^{\perp\perp} \cap V = \emptyset$.

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3. Show that $e(p)$ is open and p is compatible with every element of $e(p)$.

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Proof: Choose $R \in$ RO($\mathbb{P}$) $\setminus \{\emptyset\}$. Choose $p \in R$. $e(p) \subseteq R$, since regularization is a closure operator. Hence $e(\mathbb{P})\uparrow\ =$ RO($\mathbb{P}$) $\setminus \{\emptyset\}$. □

Lemma B. (NST, Proposition 27.21(ii), phrased differently) $(\forall p \in P)$ $e(p)$ is the largest open set U such p is compatible with every element of U. Equivalently, $e(p) = \{r \in P \mid p$ is compatible with every element of $(r]\ \}$.

Proof:

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Corollary. Let M be a c.t.m., let $\mathbb{P} \in M$ be a forcing order, and let $p \in P$. For any formula φ with all free variables among $v_0, \ldots, v_{m-1}$ and for any $\sigma_0, \ldots, \sigma_{m-1} \in M^P$,

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Eliminating G from the Forcing Theorem, continued

($\Longrightarrow$, contradiction) Now assume that $p \Vdash \varphi(\overline{\sigma})$, but $\left(p \nVdash^* \varphi(\overline{\sigma})\right)^M$.

1. $e(p) \nsubseteq [\![\varphi(\overline{\sigma})]\!]$, so $e(p) \wedge [\![\varphi(\overline{\sigma})]\!]' \neq \emptyset$.
   (Discuss BAs here.)
2. There exists $q \in P$ with $e(q) \subseteq e(p) \wedge [\![\varphi(\overline{\sigma})]\!]'$ ($\subseteq e(p)$).
   (Use Lemma A here.)
3. $e(q) \subseteq e(p)$ implies that p and q are compatible.
   (Use Lemma B here.)
4. There exists $r \in P$ with $r \leq p, q$.
5. $\left(r \Vdash^* \neg\varphi(\overline{\sigma})\right)^M$, since $e(r) \subseteq e(q) \subseteq [\![\varphi(\overline{\sigma})]\!]' = [\![\neg\varphi(\overline{\sigma})]\!]$.
6. $\exists G^{\text{generic}} \subseteq P$ containing r.
   (Discuss this here.)
7. By the Forcing Theorem, Item 5 and $r \in G$ imply $M[G] \models \neg\varphi(\overline{\sigma}_G)$.

Eliminating G from the Forcing Theorem, continued

($\Longrightarrow$, contradiction) Now assume that $p \Vdash \varphi(\overline{\sigma})$, but $\left(p \not\Vdash^* \varphi(\overline{\sigma})\right)^M$.

1. $e(p) \not\subseteq [\![\varphi(\overline{\sigma})]\!]$, so $e(p) \wedge [\![\varphi(\overline{\sigma})]\!]' \neq \emptyset$.
   (Discuss BAs here.)
2. There exists $q \in P$ with $e(q) \subseteq e(p) \wedge [\![\varphi(\overline{\sigma})]\!]'$ $(\subseteq e(p))$.
   (Use Lemma A here.)
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4. There exists $r \in P$ with $r \leq p, q$.
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Eliminating G from the Forcing Theorem, continued

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Eliminating G from the Forcing Theorem, continued

($\Longrightarrow$, contradiction) Now assume that $p \Vdash \varphi(\overline{\sigma})$, but $\left(p \not\Vdash^* \varphi(\overline{\sigma})\right)^M$.

1. $e(p) \not\subseteq [\![\varphi(\overline{\sigma})]\!]$, so $e(p) \wedge [\![\varphi(\overline{\sigma})]\!]' \neq \emptyset$.
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   (Use Lemma A here.)
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6. $\exists G^{\text{generic}} \subseteq P$ containing r.
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Eliminating G from the Forcing Theorem, continued

($\Longrightarrow$, contradiction) Now assume that $p \Vdash \varphi(\overline{\sigma})$, but $\left(p \not\Vdash^* \varphi(\overline{\sigma})\right)^M$.

1. $e(p) \not\subseteq [\![\varphi(\overline{\sigma})]\!]$, so $e(p) \wedge [\![\varphi(\overline{\sigma})]\!]' \neq \emptyset$.
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   (Use Lemma A here.)
3. $e(q) \subseteq e(p)$ implies that p and q are compatible.
   (Use Lemma B here.)
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6. $\exists G^{\text{generic}} \subseteq P$ containing r.
   (Discuss this here.)
7. By the Forcing Theorem, Item 5 and $r \in G$ imply $M[G] \models \neg\varphi(\overline{\sigma}_G)$.
8. But $M[G] \models \varphi(\overline{\sigma}_G)$, since $r \leq p$, $p \in G$.

Eliminating G from the Forcing Theorem, continued

($\Longrightarrow$, contradiction) Now assume that $p \Vdash \varphi(\overline{\sigma})$, but $\left(p \nVdash^* \varphi(\overline{\sigma})\right)^M$.

1. $e(p) \not\subseteq [\![\varphi(\overline{\sigma})]\!]$, so $e(p) \wedge [\![\varphi(\overline{\sigma})]\!]' \neq \emptyset$.
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   (Use Lemma A here.)
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6. $\exists G^{\text{generic}} \subseteq P$ containing r.
   (Discuss this here.)
7. By the Forcing Theorem, Item 5 and $r \in G$ imply $M[G] \models \neg\varphi(\overline{\sigma}_G)$.
8. But $M[G] \models \varphi(\overline{\sigma}_G)$, since $r \leq p$, $p \in G$.
9. Contradiction! $\square$

Next steps

Next steps

1. (Wednesday)

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1. (Wednesday) Use the Forcing Theorem to explain why $M[G]$ is a model.

Next steps

1. (Wednesday) Use the Forcing Theorem to explain why $M[G]$ is a model.

2. (Friday)

Next steps

1. (Wednesday) Use the Forcing Theorem to explain why $M[G]$ is a model.

2. (Friday)

Next steps

1. (Wednesday) Use the Forcing Theorem to explain why $M[G]$ is a model.
2. (Friday) Show that c.c.c. forcings preserve cardinals.