Verifying the Axiom of Comprehension

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$$(\forall \bar{p})(\forall X)(\exists Y)(\forall u)((u \in Y) \leftrightarrow ((u \in X) \wedge \varphi(u, \bar{p}))).$$

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$$(\forall u)((u \in s) \leftrightarrow (u \in t)).$$

Avoiding equality in φ

We want to construct $Z = \{(y_1, \ldots, y_n) \in X^n \mid \varphi(\bar{y})\}$ in M where φ is a formula in the language of set theory. Since we have already proved that the Axiom of Extensionality holds in M, we may replace instances of equality $(s = t)$ in φ with

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Parameter-free continuation, Connectives

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Proof: For any class C in V, let $\widehat{C}$ be the V-set of minimal-rank elements of C in V. Given $\bar{y}$, let $C_{\bar{y}}$ be the V-class $\{x \in M \mid \alpha(x, \bar{y})\}$. Thus, $\widehat{C_{\bar{y}}}$ is the V-set of minimal-rank elements of the V-class $C_{\bar{y}}$.
Given $X \in V$ with $X \subseteq M$, define $\Delta(X) = X \cup \bigcup\{\widehat{C_{\bar{y}}} \mid \bar{y} \in X^n\}$ in V. Note that $\Delta(X) \in V$ and $\Delta(X) \subseteq M$. Start at S, iterate Δ, and take the union in V: $T := \bigcup \Delta^m(S)$. The result is a set $T \in V$ such that $T \subseteq M$ and for all $\bar{y} \in T$

$$(\exists x)\ \alpha(x, \bar{y}) \to (\exists x \in T)\ \alpha(x, \bar{y}).$$

Parameter-free continuation, Existential Quantifier, Part 1

We want to construct $Z = \{(y_1, \ldots, y_n) \in X^n \mid \varphi(\bar{y})\}$ in M in the case where $\varphi(\bar{y}) = (\exists x)\alpha(x, \bar{y})$.

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Parameter-free continuation, Existential Quantifier, Part 2

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We want to construct $Z = \{\bar{y} \in X^n \mid (\exists x)\alpha(x, \bar{y})\}$.

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