Remarks about cf(κ^{card})

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Discussion. $\kappa = \aleph_1$.

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a contradiction.

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Constructing cardinals of a given cofinality

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Constructing λ-unreachable cardinals

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Let λ be an infinite cardinal. Given any cardinal μ there is a least λ-unreachable cardinal κ strictly above μ, and it is the union of (the even terms of) the sequence

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Exercise.

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Exercise. Given $\kappa < \lambda$, show how to find a λ-unreachable cardinal of cofinality κ^+.