Cardinal Numbers.

Please read LST 121-143 (Chapter 12).

Definition 1. An ordinal $\alpha \in \mathbf{On}$ is an *initial ordinal* if it is not equipotent with a smaller ordinal. An initial ordinal may also be called a *cardinal number* or a *cardinal*.

The set of natural numbers is an initial ordinal. We write ω for this set when we want to treat it as an ordinal (i.e., when the order on this set matters), and we write $\aleph_0$ for this set when we want to think of it as a cardinal number (i.e., when only the equipotence class of the set matters).

In order to determine whether there are infinite cardinals other than $\aleph_0$, we need to know how the the equipotence relation restricts to $\mathbf{On}$. The essential facts are:

(1) For each $\alpha \in \mathbf{On}$ there exists $\beta \in \mathbf{On}$ such that $|\alpha| < |\beta|$.[1]
(2) Equipotence classes of ordinals are *intervals* in the $\in$-ordering on $\mathbf{On}$. That is, the set of all ordinals equipotent to α form an interval $[\beta, \gamma)$ in $\mathbf{On}$ for some ordinals β, γ satisfying $\beta \leq \alpha < \gamma$.[2]

The two items just listed imply that the class of cardinal numbers is a proper class contained in $\mathbf{On}$. Because $\mathbf{On}$ is well-ordered, any proper subclass can be enumerated by $\mathbf{On}$. The $\aleph$-function enumerates the infinite cardinal numbers. This means that the cardinal numbers are

$$\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \aleph_0, \aleph_1, \aleph_2, \ldots.$$

That is, the set of finite ordinals equals the set of finite cardinals, and the correspondence $\alpha \mapsto \aleph_\alpha$ is an order-preserving class bijection from $\mathbf{On}$ to the class of infinite cardinal numbers. The fact that the class of cardinal numbers is a subclass of $\mathbf{On}$, the class of cardinal numbers is well-ordered by $\in$.

It is common to write α^+ to denote $S(\alpha) = \alpha \cup \{\alpha\}$ when α is an ordinal. For a cardinal κ we write κ^+ to mean $S(\kappa)$ when κ if finite, while if $\kappa = \aleph_\alpha$ we write κ^+ to mean $\aleph_{\alpha^+}$. In either case, κ^+ is the cardinal immediately succeeding κ. A cardinal κ is a *successor cardinal* if $\kappa = \lambda^+$ for some cardinal λ and κ is a *(weak) limit cardinal* if it is not $\mathbf{0}$ and not a successor cardinal.

The *cofinality* of an ordered set $\langle P; < \rangle$ is the least cardinality *cofinal* subset. (A subset $C \subseteq P$ is cofinal in $\langle P; < \rangle$ if every element of P is $\leq$ some member of C.) In particular, every cardinal has a cofinality, denoted $\mathrm{cf}(\kappa)$. A cardinal κ is *regular* if $\kappa = \mathrm{cf}(\kappa)$, otherwise it is *singular*.

Exercises.

(1) Show that $\mathrm{cf}(0) = 0$, and $\mathrm{cf}(n) = 1$ if $n \in \omega \setminus \{0\}$. (So, $0, 1$ are the only regular finite cardinals.)
(2) Show that $\mathrm{cf}(\aleph_0) = \aleph_0$, so that $\aleph_0$ is regular.
(3) Show that $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$ if α is a limit ordinal.
(4) Show that $\mathrm{cf}(\aleph_\omega) = \aleph_0$, so that $\aleph_\omega$ is singular.

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[1]This follows from Cantor's Theorem.

[2]This follows from the Cantor-Schröder-Bernstein Theorem.