# The Axioms of ZFA

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The empty set will satisfy this formula, but every atom will also satisfy the formula.

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You can see what damage it would cause if we didn't restrict Extensionality and Foundation to sets only.

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3. $V_2(A) = V_1(A) \cup \mathcal{P}(V_1(A))$ equals the union of $\{a, b\}$ and the 64-element set $\mathcal{P}(\{a, b, \emptyset, \{a\}, \{b\}, \{a, b\}\})$, which I don't want to write down. Three examples of the 'complex' elements in $V_2(A)$ are

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**Follow-up Question.**

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