The Peano Axioms.

From ~~Arithmetices principia (1889).~~ Formulario mathematico (1894).

There is a Natural Number

(1) ~~1 is a natural number~~ 0 is a natural number

Axioms of equality (= is an equivalence relation and $\mathbb{N}$ is a union of =-classes)

(2) For every natural number x, $x = x$.
(3) For all natural numbers x and y, if $x = y$, then $y = x$.
(4) For all natural numbers x, y and z, if $x = y$ and $y = z$, then $x = z$.
(5) For all a and b, if b is a natural number and $a = b$, then a is also a natural number.

Axioms of Successor

(6) For every natural number n, $S(n)$ is a natural number.
(7) For all natural numbers m and n, if $S(m) = S(n)$, then $m = n$.
(8) For every natural number n, $S(n) = 0$ is false.

Induction

(9) (Second-Order Induction)

$$\forall X(((0 \in X) \wedge (\forall x)(x \in X \rightarrow S(x) \in X)) \rightarrow (\forall x)(x \in X))$$

(9)' (First-Order Induction Scheme) For any first-order formula $\varphi(x)$

$$((\varphi(0) \wedge (\forall x)(\varphi(x) \rightarrow \varphi(S(x)))) \rightarrow (\forall x)(\varphi(x)))$$

Exercise. Work in a language whose nonlogical symbols are $0, S(x)$ and c (= an additional constant symbol). Let $\mathscr{P}$ be the set of **P1-P9**. Let $\Gamma = \mathscr{P} \cup \Sigma$ where Σ contains all sentences of the form $c \neq 0, c \neq S(0), c \neq S(S(0)), \ldots$. A model of Γ is a Peano structure containing a nonstandard natural number c.

(1) Show that Γ is finitely satisfiable.
(2) Explain why Γ has a model if the first-order version of **P9** is used, but does not have a model if the second-order version of **P9** is used.