Permutation Models of ZFA

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To show that $\mathfrak{M}$ is almost universal, choose $x \in V$ such that $x \subseteq \mathfrak{M}$. We must locate $y \in \mathfrak{M}$ such that $x \subseteq y$. Since $x \in V$ and $x \subseteq \mathfrak{M} \subseteq V(A)$, we have $x \subseteq V_\alpha(A) \cap \mathfrak{M}$ for some α. (Here I am using the fact that $\text{rank}_{V(A)}(x) \leq \text{rank}_V(x)$.) I will take $y = V_\alpha(A) \cap \mathfrak{M}$. Each $z \in y$ belongs to $\mathfrak{M}$, hence is hereditarily symmetric. If we show that y itself is symmetric, then y will be hereditarily symmetric. Since $\mathfrak{M}$ is symmetric, it suffices to verify that $V_\alpha(A)$ is symmetric. In fact,

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$\mathfrak{M}$ Satisfies ZFA Extensionality.

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$\mathfrak{M}$ **Satisfies ZFA Extensionality.** Assume that $x, y \in \mathfrak{M}$ are sets (i.e., $x, y \notin A$)

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$\mathfrak{M}$ **Satisfies ZFA Extensionality.** Assume that $x, y \in \mathfrak{M}$ are sets (i.e., $x, y \notin A$) and x and y have the same elements.

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$\mathfrak{M}$ **Satisfies ZFA Foundation.** Assume that x is a set in $\mathfrak{M}$. (Not an atom!) Then x has the same elements and the same $\in$-relation in V, where Foundation holds. Let $y \in x$ be $\in$-minimal in V. ($y \in x$ and $y \cap x = \emptyset$). Then $y \in \mathfrak{M}$ (by transitivity), $y \in x$, and $y \cap x = \emptyset$ in $\mathfrak{M}$.