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For the complete proof, see NST Theorem 28.24, pages 609-611.

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Claim. $\overline{\alpha}_G$ contains all elements of the elements of α_G $(= X)$.

Assume that $\gamma_G \in \beta_G \in \alpha_G$. Then $\gamma, \beta \in M^P$ and $\exists p, q \in G$ such that $(\gamma, q) \in \beta$ and $(\beta, p) \in \alpha$. We have $\beta \in \mathrm{dom}(\alpha)$, so $(\gamma, q) \in \overline{\alpha}$.

Proof of the theorem: Union (weak version)

Choose $X \in M[G]$. Choose $\alpha \in M^P$ such that $\alpha_G = X$.
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Proof of the theorem: Union (weak version)

Choose $X \in M[G]$. Choose $\alpha \in M^P$ such that $\alpha_G = X$.
Example: $\alpha = \{(\beta, p), (\gamma, q), \ldots\}$.
Since M is a c.t.m., and $\alpha \in M^P \subseteq M$, $\mathrm{dom}(\alpha) \in M$, hence $\bigcup \mathrm{dom}(\alpha) \in M$. Let $\overline{\alpha} = \bigcup \mathrm{dom}(\alpha)$.
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Comprehension will complete the argument.

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1. Well order $\mathrm{dom}(\alpha) \in M$ with a bijection $f\colon \kappa^{\text{card}} \to \mathrm{dom}(\alpha)$.
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5. κ is totally ordered by $\in$ in M, hence V, hence $M[G]$.
6. κ is well ordered in $M[G]$.
7. Well order α_G be $\gamma_G \mapsto$ least $\lambda \in \kappa$ such that $f(\lambda)_G = \gamma_G$.