The Construction of $M[G]$

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Proof: 10 lines, pages 199-200, NST.

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Call τ a P**-name** if $F_P(\tau) = 1$. By the preceding claims,

P-names (page 599, NST)

Theorem 28.4, NST. For any set P there exists a unique class function $F = F_P \colon \mathbf{V} \to 2$ such that for any set τ,

$$F(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is a binary relation and} \\ & \text{for all } (\sigma, p) \in \tau \text{ we have that } p \in P \text{ and } F(\sigma) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This theorem uses recursion over the class $A = V$ using the well-founded, set-like relation E defined by

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Characterizing P-names

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Claim. τ is a P-name iff τ is a binary relation such that for every $(\sigma, p) \in \tau$ it is the case that σ is a P-name and $p \in P$.

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$$\tau_G = \{\sigma_G \mid \sigma \in M^P, (\exists p \in G)((\sigma, p) \in \tau)\}$$

for all $\tau \in M^P$ when M is a c.t.m. and $G \subseteq P \in M$. (Slide 9!) Hence

$$\begin{aligned} \text{rank}(\tau_G) &= \bigcup\{\text{rank}(\sigma_G) + 1 \mid (\exists p)(\sigma \in (\sigma, p) \in \tau)\} \\ &\leq \bigcup\{\text{rank}(\sigma) + 1 \mid (\exists p)(\sigma \in (\sigma, p) \in \tau)\} \qquad \leq \text{rank}(\tau). \end{aligned}$$

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3. $\kappa \leq 2^{\omega}$ in $M[G]$. (Completed, assuming the above parts.)