

Logic

Example

Newton's Third Law of Motion declares:

For every action, there is an equal and opposite reaction.

Formally, we might write this as:

$$(\forall A)(\exists R)(\text{Equal}(A, R) \wedge \text{Opposite}(A, R))$$

This is a fairly typical structure for a declarative sentence. In mathematics, we may encounter sentences with similar structure. We might want to say that the ordered set $\langle \mathbb{R}; < \rangle$ does not have a least or largest element. We could express this by saying that $\langle \mathbb{R}; < \rangle$ satisfies the sentence:

$$(\forall x)(\exists y)(\exists z)((y < x) \wedge (x < z))$$

Next Goals

We are going to spend some time learning about logic. The subject of logic is concerned with the correctness of reasoning. We shall discuss this topic in stages, by addressing the following questions.

- 1 What kinds of mathematical objects do we want to study and write about?
- 2 What is the syntax of a correctly written statement?
- 3 What does it mean for a statement to be true?
- 4 What is a proof?
- 5 What is the difference between truth and provability?
- 6 What are some strategies for creating and writing proofs?

Associated Reading:

- 1 Chapter 3. Propositional Logic.
(Subsections 3.1, 3.5.1, 3.6.1 only)
 - 1 Discusses the use of the logical connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$.
- 2 Chapter 4. First-order Logic.
(Subsections 4.1.1, 4.1.2, 4.1.3, 4.2.1, 4.2.3, 4.3.2 only)
 - 1 Discusses the use of the quantifiers $\forall$, $\exists$ and the use of predicates.

Example: Goldbach's Conjecture

Goldbach's Conjecture is the statement that every even number greater than 2 is a sum of two prime numbers. Initial evidence:

$$4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 5 + 5, \dots, 4 \times 10^{18} = ? + ? \dots$$

No one knows if Goldbach's Conjecture is true, which is why it is called a "Conjecture". Let us write it down formally. Since the statement refers to even numbers, it will help to have a formal expression $\varphi_{\text{even}}(x)$ which is true about x exactly when x is even. Similarly, it will help to have a formal expression $\varphi_{\text{prime}}(x)$ which is true about x exactly when x is prime. If we had such expressions, then Goldbach's Conjecture could be written

$$(\forall x)((x > 2) \wedge \varphi_{\text{even}}(x)) \rightarrow (\exists y)(\exists z)(\varphi_{\text{prime}}(y) \wedge \varphi_{\text{prime}}(z) \wedge (x = y + z)).$$

I would read this aloud by saying "For all x , if x is greater than 2 and satisfies a formula expressing that x is even, then there exists y and z , each satisfying a formula expressing that they are prime, such that x equals y plus z ".

Goldbach's Conjecture, 2

In this formal statement $\varphi_{\text{even}}(x)$ is an abbreviation for something like

$$(\exists w)(x = w + w),$$

Which I would read as “There exists w such that x equals w plus w ”.

$\varphi_{\text{prime}}(x)$ is an abbreviation for something like

$$(x > 1) \wedge (\forall u)((\exists v)(x = uv) \rightarrow ((u = 1) \vee (u = x))),$$

which I would read as “ x is greater than 1 and for all u , if there exists v such that $x = uv$, then u equals 1 or x ”.

One purpose for expressing Goldbach's Conjecture formally is to make the underlying structure of the sentence clear.