A Theorem for Checking the Axioms

The theorem to be proved

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I may refer to V as "the universe".

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Note: I will save Comprehension until the end, since it is the most difficult to verify.

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Proof that the Axiom of Foundation holds

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Assume that $x \in M$ is nonempty. The element x is also nonempty as an element of V. Apply the Axiom of Foundation in V to obtain an $\in$-minimal element $z \in x$.

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Assume that $x \in M$ is nonempty. The element x is also nonempty as an element of V. Apply the Axiom of Foundation in V to obtain an $\in$-minimal element $z \in x$. The claim that z is an $\in$-minimal element of x combines the claims that (i) $z \in x$ and (ii) $z \cap x = \emptyset$.

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The universe V contains an inductive set, I. The set I contains $\emptyset$ and is closed under successor.

Proof that the Axiom of Infinity holds

We will verify a weak form of the Axiom of Infinity, which will reduce to the full axiom provided the Axiom of Comprehension holds.

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Goal. Argue that, if Comprehension holds, then it is possible to separate an inductive set from y.

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Let F be a class function relative to M. (That is, the formula that says that F satisfies the function rule holds in $\langle M; \in \rangle$.)

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$F^V[A] \in V$. $F^V[A]$ contains each $F(a)$, $a \in A$. Since each $F(a) \in M$ and M is almost universal, there is a $y \in M$ such that $F^V[A] \subseteq y$.

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is a set in M. The proof for this axiom is accomplished by induction on the complexity of the formula φ. Recall that formulas are defined by recursion as follows: The set of all first-order formulas whose only nonlogical symbol is $\in$ is the smallest set $\mathcal{F}$ such that

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We proceed by induction on the complexity of the defining formula, using the Gödel operations to show the the required sets must be member of M.