A Theorem for Checking the Axioms

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I may refer to V as "the universe".

Definitions.

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Note: I will save Comprehension until the end, since it is the most difficult to verify.

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Construct \emptyset by applying Comprehension to the set from Claim 1.

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Construct \emptyset by applying Comprehension to the set from Claim 1. Construct S(x) using pairing and union.

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The universe V contains an inductive set, I. The set I contains \emptyset and is closed under successor. We want to show that M has a set with these properties.

Claim 1. $M \neq \emptyset$.

Since M is almost universal, it must contain a set. (Choose $\emptyset \in V$. Since $\emptyset \subseteq M$, there must exist $y \in M$ such that $\emptyset \subseteq y$. This $y \in M$ witnesses that $M \neq \emptyset$.)

Claim 2. $\emptyset \in M$ and the successor operation $S(x) = x \cup \{x\}$ is a class function on M.

Construct \emptyset by applying Comprehension to the set from Claim 1. Construct S(x) using pairing and union.



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Goal.

Choose $I=\omega^V$. Let $J=I\cap M$. J is a set in V that consists of the finite ordinals in V that lie in M. Since M is almost universal, there is a $y\in M$ such that $J\subseteq y$.

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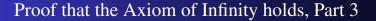
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Proof that the Axiom of Replacement holds

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 $F^V[A] \in V$. $F^V[A]$ contains each F(a), $a \in A$. Since each $F(a) \in M$ and M is almost universal, there is a $y \in M$ such that $F^V[A] \subseteq y$.

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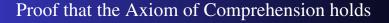
 $F^V[A] \in V$. $F^V[A]$ contains each F(a), $a \in A$. Since each $F(a) \in M$ and M is almost universal, there is a $y \in M$ such that $F^V[A] \subseteq y$. Construct $F^M[A]$ from y by Comprehension using the property "I am in the image of F".

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We proceed by induction on the complexity of the defining formula, using the Gödel operations to show the the required sets must be member of M.