Fraenkel's First Model of ZFA

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$V(A)$ is the class of all x for which there exists $\alpha \in \mathrm{ON}$ with $x \in V_\alpha(A)$.

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There are first-order formulas $\varphi_{\text{atom}}(x)$, $\varphi_{\text{pure set}}(x)$, $\varphi_{\text{ordinal}}(x)$ in the language whose nonlogical symbols are $\in$, A, 0 that define each of these types of sets.

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Fraenkel's First model is the permutation model $\mathfrak{M}$ associated to this choice for $A, G, \mathcal{F}$.

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Now, assume that $\gamma\colon D \to \bigcup D$ $(= A)$ is a choice function for D in the model $\mathfrak{M}$. Since $\gamma \in \mathfrak{M}$, there is a finite set $\{a_1, \ldots, a_m\}$ such that

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Choose distinct $p, q \in A - \{a_1, \ldots, a_m\}$. Label them so that $\gamma(\{p, q\}) = p$. Since $\{a_1, \ldots, a_m\} \cap \{p, q\} = \emptyset$, the transposition $\sigma = (p\ q) \in G$ stabilizes $\{a_1, \ldots, a_m\}$, so $\sigma \in G_{a_1,\ldots,a_m}$.

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