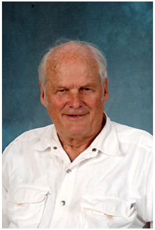
The “Main Theorem of Cardinal Arithmetic”



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Let κ and λ be cardinals with $2 \leq \kappa$ and λ infinite.

(1) *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*

Henceforth assume $\lambda < \kappa$. In particular, κ is infinite.

(2) *If κ is λ-reachable*[a] *from some $\mu < \kappa$, then $\kappa^\lambda = \mu^\lambda$.*

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Henceforth assume that κ is λ-unreachable from below.

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Henceforth assume that κ is λ-unreachable from below.

(3) (a) *if $\lambda < cf(\kappa)$, then $\kappa^\lambda = \kappa$.*
(b) *if $cf(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^{cf(\kappa)}$.*

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$$\underset{\uparrow}{2}^\lambda \leq \underset{\uparrow}{\kappa}^\lambda \leq \lambda^\lambda \leq (2^\lambda)^\lambda$$

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$$\begin{aligned} \underset{\uparrow}{\kappa} \leq \underset{\uparrow}{\kappa}^\lambda &= \left|\bigcup_{i<\text{cf}(\kappa)} \alpha_i^\lambda\right| \leq \left|\bigsqcup_{i<\text{cf}(\kappa)} \alpha_i^\lambda\right| \\ &= \sum_{i<\text{cf}(\kappa)} |\alpha_i|^\lambda \end{aligned}$$

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for some $i < \text{cf}(\kappa)$. This implies that $\kappa^\lambda = \bigcup_{i<\text{cf}(\kappa)} \alpha_i^\lambda$. Hence

$$\begin{aligned} \underset{\uparrow}{\kappa} \leq \underset{\uparrow}{\kappa^\lambda} &= \left|\bigcup_{i<\text{cf}(\kappa)} \alpha_i^\lambda\right| \leq \left|\bigsqcup_{i<\text{cf}(\kappa)} \alpha_i^\lambda\right| \\ &= \sum_{i<\text{cf}(\kappa)} |\alpha_i|^\lambda \leq \sum_{i<\text{cf}(\kappa)} \kappa \leq \text{cf}(\kappa) \cdot \kappa = \underset{\uparrow}{\kappa}. \end{aligned}$$

Proof of (3)(a)

Assumptions:

- $\lambda^{\text{inf}} < \text{cf}(\kappa) \ \ (\leq \kappa)$.
- κ is λ-unreachable from below.

Conclusion: $\kappa^\lambda = \kappa$.

Choose a sequence $\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \kappa$ of length $\text{cf}(\kappa)$ such that $\bigcup_{i<\text{cf}(\kappa)} \alpha_i = \kappa$. Any function $f \in \kappa^\lambda$ has image of size $|f[\lambda]| \leq \lambda < \text{cf}(\kappa)$, contained in κ, hence $f[\lambda]$ is not cofinal in κ. It follows that $f[\lambda]$ is bounded above by some α_i. This implies that

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$\square$

The α-truncation of $f\colon \lambda \to \kappa$

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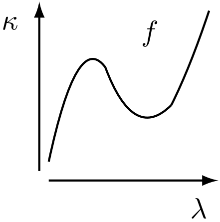

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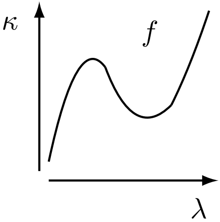
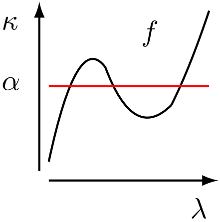
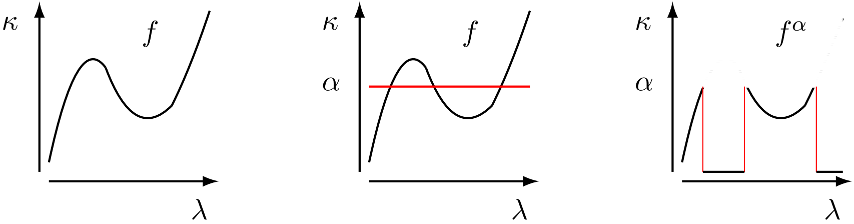




The α-truncation of $f : \lambda \to \kappa$



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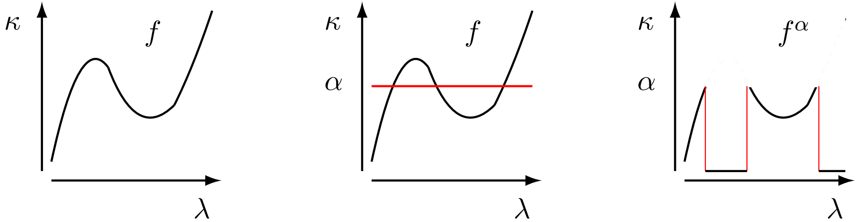

$$f^{\alpha}(x) = \begin{cases} f(x) & \text{if } f(x) < \alpha \\ 0 & \text{else.} \end{cases}$$

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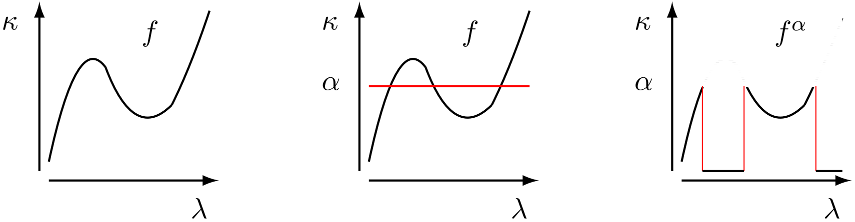

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The α-truncation of f "reveals" the part of the graph of f bounded above by the line $y = \alpha$.

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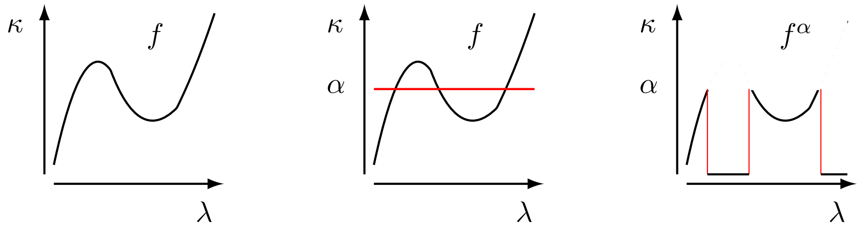

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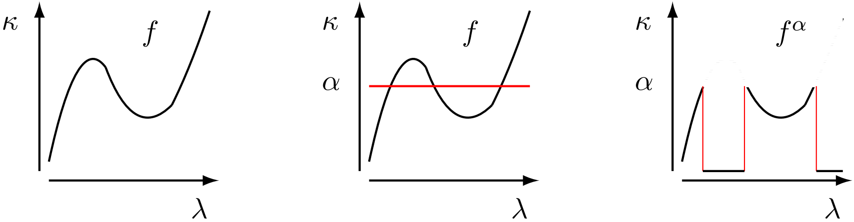
The α-truncation of $f : \lambda \to \kappa$



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$$f \in \kappa^\lambda \quad \Longrightarrow \quad f^\alpha \in \alpha^\lambda.$$

Proof of (3)(b)

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Conclusion: $\kappa^{\lambda} = \kappa^{\mathrm{cf}(\kappa)}$.

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