c.c.c. forcing preserves cardinals

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- In M, there is a function $F \colon \alpha \to \mathcal{P}(\lambda)$ such that for $a \in \alpha$ we have $f(a) \in F(a)$ and $|F(a)|$ countably infinite.
- $S = \bigcup_{a \in \alpha} F(a)$ is cofinal in λ in M, but smaller than $\lambda = \mathrm{cf}^{M}(\lambda)$.

Stage 3

Theorem. Let M be a c.t.m. and let $\mathbb{P} \in M$ be a c.c.c. forcing order.

1. $\mathbb{P}$ preserves infinite regular cardinals.
2. $\mathbb{P}$ preserves cofinalities of limit ordinals.
3. $\mathbb{P}$ preserves cardinals.

Proof: It suffices to prove the first statement, then cite the Stage 1 Theorem.

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