Stirling numbers and Bell numbers

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There are many parallels between $C(n,k)$ and $S(n,k)$.

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where $x^{\underline{k}} = (x)_k = x(x-1)\cdots(x-(k-1))$.

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$n\backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
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1	0	1	0	0	0	0	0	0	0	$\cdots$
2	0	1	1	0	0	0	0	0	0	$\cdots$
3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
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The n row sum is denoted B_n and is called the nth **Bell number**.

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Proof. Let $X = \{x_1, x_2, \ldots, x_{n+1}\}$.

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Another interesting relation is $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot B_k$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+1}\}$. A partition of X is determined by the choice of the cell $[x_{n+1}]$ (= a subset of X containing x_{n+1}) and a partition of $X - [x_{n+1}]$.

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

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- $n^n \leq 2^{n^2}$, since the latter counts the number of binary relations from n to n, while the former only counts the binary relations that are functions.