Stirling numbers of the second kind!

Definition 1. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad \mathrm{S}(n,k),$$

and is called a *Stirling number of the second kind.*

Theorem 2. *(Formula for Stirling numbers.)*
$S(n,k) = \frac{1}{k!}\left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n\right)$.

Theorem 3. *(Recursion for Stirling numbers.)*

(1) $\mathrm{S}(n,k) = 0$ *if* $n < k$.
(2) $\mathrm{S}(0,0) = 1$ *and* $\mathrm{S}(n,0) = 0$ *if* $n > 0$.
(3) $\mathrm{S}(n,k) = \mathrm{S}(n-1,k-1) + k \cdot \mathrm{S}(n-1,k)$

Theorem 4. *(Binomial-type Theorem for Stirling Numbers.)*
$x^n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}}$, *where* $x^{\underline{k}} = (x)_k = x(x-1)\cdots(x-(k-1))$.

Definition 5. The total number of partitions of an n-element set is denoted B_n, and is called the nth *Bell number.*

Theorem 6. $B_n = \sum_{k=0}^{n} S(n,k)$.

Problems.

(1) Determine how the numbers $2^{n-1}, B_n, n!, n^n, 2^{n^2}$ are related to each other as n grows. (Which is larger than which?)

(2) Show that the Bell numbers are equal to the sequence defined recursively by

$$\begin{aligned} B_0 &= 1 \\ B_{n+1} &= \binom{n}{n} B_n + \binom{n}{n-1} B_{n-1} + \cdots + \binom{n}{0} B_0. \end{aligned}$$

(3) Show that $S(n,n-1) = C(n,2)$. Find a formula involving binomial coefficients for $S(n,n-2)$.

(4) Show that if p is prime, then p divides $S(p,k)$ whenever $1 < k < p$. (Hint: imagine cyclically permuting the numbers $1, 2, \ldots, p$. Show that partitions into k-cells get permuted in p-cycles.)

(5) Suppose that $|A| = n$ and $|B| = m$. How many pairs (X,Y) are there where $X = \mathrm{coim}(f)$ and $Y = \mathrm{im}(f)$ for some function $f\colon A \to B$?

$n\backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1	0	0	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	0	0	$\cdots$
2	0	1	1	0	0	0	0	0	0	$\cdots$
3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$