Practice with Inclusion/Exclusion, Stirling, and Bell numbers!

(1) Let $m = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 = 969969$ be the product of the first 8 distinct prime numbers. How many ways are there to factor m? Here, a factorization of m is a representation of m as a product of natural numbers greater than 1, as in $m = 30 \cdot 323 \cdot 1001$. (Assume that the order of the factors does not matter, so $m = 30 \cdot 323 \cdot 1001$ and $m = 1001 \cdot 323 \cdot 30$ are the same factorization.)

Factorizations of m (like $m = 30 \cdot 323 \cdot 1001 = (2 \cdot 3 \cdot 5)(17 \cdot 19)(7 \cdot 11 \cdot 13)$) correspond to partitions of $\{2, 3, 5, 7, 11, 13, 17, 19\}$ (like $2, 3, 5/17, 19/7, 11, 13$). The number of partitions of this 8-element set of primes is $B_8 = 4140$.

(2) In a class of 20 students, how many study groups can be formed which include at least one of the three students Archibald, Beryl, or Cornelia? Assume that a study group must involve at least 2 students.

Let A be the set of study groups that contain Archibald, B be the set of study groups that contain Beryl, and C be the set of study groups that contain Cornelia. We want to compute $|A \cup B \cup C|$.

It is not hard to see that the number of study groups that contain Archibald is $2^{20-1} - 1$. (The non-Archibald members of the study group form a nonempty subset chosen from the $20 - 1$ other students.) Similarly, the number of study groups that contain Archibald and Beryl is $2^{20-2} = 2^{18}$, and the number of study groups that contain Archibald, Beryl, and Cornelia is $2^{20-3} = 2^{17}$. By Inclusion/Exclusion

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 3|A| - 3|A \cap B| + |A \cap B \cap C| \\ &= 3(2^{19} - 1) - 3(2^{18}) + 2^{17}. \end{aligned}$$

(3) How many 6-digit numbers have the property that, for every k, the kth digit is different than the $(7-k)$th digit?

Let X be the set of all 6-digit numbers. Let A be the subset of X consisting of numbers whose first and last digit are equal. Let B be the subset of X consisting of numbers whose second and second-to-last digit are equal. Let C be the subset of X consisting of numbers whose third and third-to-last digit are equal. We are trying to count $|X| - |A \cup B \cup C|$:

$$\binom{3}{0}10^6 - \binom{3}{1}10^5 + \binom{3}{2}10^4 - \binom{3}{3}10^3 = 10^3 9^3 = 729000.$$

Another way to solve this problem is to pick the first three digits arbitrarily (in $10 \times 10 \times 10$ ways), then pick the last three so that they satisfy the conditions that the 1st digit is different from the 6th, the 2nd is different from the 5th, and the 3rd is different from the 4th ($9 \times 9 \times 9$ ways to pick the last 3 digits). Altogether this yields $10^3 9^3$ numbers.

(4) A news organization reports that the percentage of voters who would be satisfied with candidates A, B, C for President is 65%, 57%, 58% respectively. Furthermore, 28% would accept A or B, 30% would accept A or C, 27% would accept B or C, and 12% would accept any of the three. Is this fake news?

Yes, it is fake news. One calculates that 107% of voters support at least one candidate.

(5) If $f : k \to k$ is a bijection, then i is called a fixed point of f if $f(i) = i$. What percentage of bijections $f : k \to k$ have no fixed points? (Count the number of bijections with no fixed points, then divide by the total number of of bijections.)

Let X be the set of all bijections $f\colon k \to k$. $|X| = k!$. Now, for $i = 1 \ldots, k$, let A_i be the subset of X consisting of the bijections where $f(i) = i$. We want to first count $d = |X| - |A_1 \cup \cdots \cup A_k|$, and then compute $d/|X| = d/k!$. The value of d is

$$\begin{aligned} d &= k! - \left(\binom{k}{1}(k-1)! - \binom{k}{2}(k-2)! + \cdots\right) \\ &= k!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!}\right). \end{aligned}$$

Thus $d/k! = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!}$. Observe that this percentage approaches $\frac{1}{e}$ as $k \to \infty$.

Remarks: a permutation with no fixed points is called a "derangement". This exercise shows that the percentage of permutations of a k-element set that are derangements is approximately $\frac{1}{e}$ when k is large.

(6) Explain why $S(n, 2) = 2^{n-1} - 1$ if $n > 0$.

$S(n, 2)$ is the number of partitions of $X = \{1, 2, \ldots, n\}$ into 2 cells. When $X \neq \emptyset$ ($n > 0$), such a partition will look like $1ab \cdots m/pqr \cdots z$, with the first cell inhabited by the element 1 and possibly some other elements and the second cell inhabited by some nonempty subset of X. These partitions can be counted by specifying the elements in the second cell. The set of elements of the second cell can be any nonempty subset of the set $X \setminus \{1\} = \{2, 3, \ldots, n\}$. Thus, $S(n, 2)$ equals the number of nonempty substes of an $(n-1)$-element set, which is $2^{n-1} - 1$.

(7) Explain why $S(n, n-1) = \binom{n}{2}$.

$S(n, n-1)$ is the number of partitions of $\{1, 2, \ldots, n\}$ into $n-1$ cells. All the cells must have one element, except that one cell has two elements. We can count partitions like this by first choosing the elements which inhabit the 2-element cell and then allowing all other elements to inhabit singleton cells. We can choose the 2-element cell in $\binom{n}{2}$ ways.

(8) Assume that $|A| = 10$. How many equivalence relations on A have 5 equivalence classes?

$S(10, 5) = 42525$.

(9) How many solutions are there to $x_1 + x_2 + x_3 + x_4 = 25$ if each x_i must be a natural number from the interval $[0, 10]$?

Let X be the set of solutions to $x_1 + x_2 + x_3 + x_4 = 25$ in the nonnegative integers. Let A_i be the subset of X where $x_i > 10$. We want to count $|X| - |A_1 \cup A_2 \cup A_3 \cup A_4|$. The answer is

$$\binom{4+25-1}{25} - \left(\binom{4}{1}\binom{4+14-1}{14} - \binom{4}{2}\binom{4+3-1}{3}\right).$$