Practice with abbreviations!

Write the following statements formally. In each case, draw a formula tree for the statement.

(1) The Axiom of Extensionality.

(2) $\varphi_{x=\emptyset}(x)$, which expresses "x has no elements". Then express the Axiom of the Empty Set.

(3) $\varphi_{(p=\{x,y\})}(x, y, p)$, which expresses "$p = \{x, y\}$". Then express the Axiom of Pairing.

(4) $\varphi_{(y=S(x))}(x, y)$, which expresses "y is the successor of x".

(5) $\varphi_{\mathrm{ind}}(I)$, which expresses that I is an inductive set.

(6) Express the Axiom of Infinity.

(7) $\varphi_{(y=\bigcup x)}(x, y)$, which expresses that y is the union of x. Then express the Axiom of Union.

Some answers!

(1) $(\forall x)(\forall y)((x = y) \leftrightarrow \forall z((z \in x) \leftrightarrow (z \in y)))$. (Tree?)

(2) $\varphi_{x=\emptyset}(x)$ could be

$$\neg(\exists y)(y \in x) \qquad \text{or} \qquad (\forall y)(\neg(y \in x)) \qquad \text{or} \qquad (\forall y)(y \notin x).$$

To express the Axiom of the Empty Set, write $(\exists x)\varphi_{x=\emptyset}(x)$ or $(\exists x)(\forall y)(\neg(y \in x))$. (Tree?)

(3) $\varphi_{(p=\{x,y\})}(x, y, p)$ could be

$$(\forall z)((z \in p) \leftrightarrow ((z = x) \vee (z = y))).$$

The Axiom of Pairing could be

$$(\forall x)(\forall y)(\exists p)\varphi_{(p=\{x,y\})}(x, y, p),$$

or

$$(\forall x)(\forall y)(\exists p)(\forall z)((z \in p) \leftrightarrow ((z = x) \vee (z = y))).$$

(4) $\varphi_{(y=S(x))}(x, y)$ could be

$$(\forall z)((z \in y) \leftrightarrow (z \in x) \vee (z = x)).$$

(5) $\varphi_{\text{ind}}(I)$ could be

$$(\exists x)((x \in I) \wedge (\varphi_{x=\emptyset}(x)) \wedge (\forall y)((y \in I) \to (\exists z)((z \in I) \wedge \varphi_{(z=S(y))}(y, z))).$$

(6) The Axiom of Infinity could be expressed

$$(\exists I)\varphi_{\text{ind}}(I).$$

(Try expanding this so that it uses no abbreviations!)

(7) $\varphi_{(y=\bigcup x)}(x, y)$ could be

$$(\forall z)((z \in y) \leftrightarrow (\exists w)((z \in w) \wedge (w \in x))).$$

The Axiom of Union could be

$$(\forall x)(\exists y)\varphi_{(y=\bigcup x)}(x, y).$$