The Natural Numbers

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$$\varphi_{y=S(x)}(x, y): \quad (\forall z)((z \in y) \leftrightarrow ((z \in x) \vee (z = x))).$$

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This axiom guarantees that $\mathbb{N}$ exists.

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Proof. Recall that we have defined $\mathbb{N}$ so that it is the intersection of all inductive sets, say $\mathbb{N} = \bigcap_{I \text{ inductive}} I$. To prove that $\mathbb{N}$ is inductive, we must show that it contains 0 and it is closed under successor.

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