The Natural Numbers

$\mathbb{N}$

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets.

$\mathbb{N}$

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets.
We will see that $\mathbb{N}$ is totally ordered by the $\in$-relation:

$$m < n \qquad \Leftrightarrow \qquad m \in n.$$

$\mathbb{N}$

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets.
We will see that $\mathbb{N}$ is totally ordered by the $\in$-relation:

$$m < n \qquad \Leftrightarrow \qquad m \in n.$$

We will also see that $\mathbb{N}$ may be equipped with algebraic structure:

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets.
We will see that $\mathbb{N}$ is totally ordered by the $\in$-relation:

$$m < n \qquad \Leftrightarrow \qquad m \in n.$$

We will also see that $\mathbb{N}$ may be equipped with algebraic structure:
Addition

$$\begin{aligned} m + 0 &:= m && \text{(IC)} \\ m + S(n) &:= S(m+n) && \text{(RR)} \end{aligned}$$

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets.
We will see that $\mathbb{N}$ is totally ordered by the $\in$-relation:

$$m < n \qquad \Leftrightarrow \qquad m \in n.$$

We will also see that $\mathbb{N}$ may be equipped with algebraic structure:
Addition

$$\begin{aligned} m + 0 &:= m && \text{(IC)} \\ m + S(n) &:= S(m+n) && \text{(RR)} \end{aligned}$$

Multiplication

$$\begin{aligned} m \cdot 0 &:= 0 && \text{(IC)} \\ m \cdot S(n) &:= (m \cdot n) + m && \text{(RR)} \end{aligned}$$

The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets.
We will see that $\mathbb{N}$ is totally ordered by the $\in$-relation:

$$m < n \qquad \Leftrightarrow \qquad m \in n.$$

We will also see that $\mathbb{N}$ may be equipped with algebraic structure:
Addition

$$\begin{aligned} m + 0 &:= m && \text{(IC)} \\ m + S(n) &:= S(m+n) && \text{(RR)} \end{aligned}$$

Multiplication

$$\begin{aligned} m \cdot 0 &:= 0 && \text{(IC)} \\ m \cdot S(n) &:= (m \cdot n) + m && \text{(RR)} \end{aligned}$$

Exponentiation

$$\begin{aligned} m^0 &:= 1 && \text{(IC)} \\ m^{S(n)} &:= (m^n) \cdot m && \text{(RR)} \end{aligned}$$

Induction and Recursion

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction and Recursion

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)

Induction and Recursion

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \mid \varphi(x)\}$ is subset of $\mathbb{N}$.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \,|\varphi(x)\}$ is subset of $\mathbb{N}$. If the two conditions of the theorem hold,

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \mid \varphi(x)\}$ is subset of $\mathbb{N}$. If the two conditions of the theorem hold, then I is an inductive subset of $\mathbb{N}$.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \,|\varphi(x)\}$ is subset of $\mathbb{N}$. If the two conditions of the theorem hold, then I is an inductive subset of $\mathbb{N}$. Hence $I = \mathbb{N}$.

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \mid \varphi(x)\}$ is subset of $\mathbb{N}$. If the two conditions of the theorem hold, then I is an inductive subset of $\mathbb{N}$. Hence $I = \mathbb{N}$. (Since $I \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq I$.)

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \mid \varphi(x)\}$ is subset of $\mathbb{N}$. If the two conditions of the theorem hold, then I is an inductive subset of $\mathbb{N}$. Hence $I = \mathbb{N}$. (Since $I \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq I$.) $\square$

To develop the structure $\langle \mathbb{N}; 0, S(x), x < y, x + y, x \cdot y, x^y \rangle$, we need the tools of Induction and Recursion.

Induction.

Theorem. (Principle of Induction)
Let $\varphi(x)$ be a (first-order ZFC-) formula. If

1. $\varphi(0)$ is true, and
2. $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in N$, then

$\varphi(n)$ is true for all $n \in \mathbb{N}$.

Proof. By the Axiom of Separation, $I = \{x \in \mathbb{N} \mid \varphi(x)\}$ is subset of $\mathbb{N}$. If the two conditions of the theorem hold, then I is an inductive subset of $\mathbb{N}$. Hence $I = \mathbb{N}$. (Since $I \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq I$.) $\square$

Example

Example

Suppose we observe that

Example

Suppose we observe that

$$1 = 1$$

Example

Suppose we observe that

$$\begin{aligned} 1 &= 1 \\ 1 + 3 &= 4 \end{aligned}$$

Example

Suppose we observe that

$$\begin{array}{rcrcrcr} & & & & 1 & = & 1 \\ & & 1 & + & 3 & = & 4 \\ 1 & + & 3 & + & 5 & = & 9 \end{array}$$

Example

Suppose we observe that

$$\begin{array}{rcrcrcrcl} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

Example

Suppose we observe that

$$\begin{array}{rcrcrcrcl} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2.

Example

Suppose we observe that

$$\begin{array}{rcrcrcrcl} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

Example

Suppose we observe that

$$\begin{array}{ccccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

Example

Suppose we observe that

$$\begin{array}{rcrcrcrcl} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture.

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction")

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

Example

Suppose we observe that

$$\begin{array}{ccccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

1. (Basis of Induction) $\varphi(0)$ is true.

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the “Principle of Induction”) by proving:

1. (Basis of Induction) $\varphi(0)$ is true.

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

1. (Basis of Induction) $\varphi(0)$ is true. (Check!)

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n + 1) = (n + 1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

1. (Basis of Induction) $\varphi(0)$ is true. (Check!)
2. (Inductive Step) $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in \mathbb{N}$.

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

1. (Basis of Induction) $\varphi(0)$ is true. (Check!)
2. (Inductive Step) $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in \mathbb{N}$.

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

1. (Basis of Induction) $\varphi(0)$ is true. (Check!)
2. (Inductive Step) $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in \mathbb{N}$. (Check!)

Example

Suppose we observe that

$$\begin{array}{ccccccccc} & & & & & & 1 & = & 1 \\ & & & & 1 & + & 3 & = & 4 \\ & & 1 & + & 3 & + & 5 & = & 9 \\ 1 & + & 3 & + & 5 & + & 7 & = & 16 \end{array}$$

and we conjecture that the sum of the first n odd numbers is n^2. Suppose we consider

$$\varphi(n) : 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$$

to be a formula that expresses this conjecture. Then we could establish the conjecture (using the "Principle of Induction") by proving:

1. (Basis of Induction) $\varphi(0)$ is true. (Check!)
2. (Inductive Step) $\varphi(k)$ implies $\varphi(S(k))$ is true for all $k \in \mathbb{N}$. (Check!)